Chiral Hall Effect in Noncollinear Magnets from a Cyclic Cohomology Approach

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We demonstrate the emergence of an anomalous Hall effect in chiral magnetic textures which is neither proportional to the net magnetization nor to the well-known emergent magnetic field that is responsible for the topological Hall effect. Instead, it appears already at linear order in the gradients of the magnetization texture and exists for one-dimensional magnetic textures such as domain walls and spin spirals. It receives a natural interpretation in the language of Alain Connes' noncommutative geometry. We show that this chiral Hall effect resembles the familiar topological Hall effect in essential properties while its phenomenology is distinctly different. Our findings make the reinterpretation of experimental data necessary, and offer an exciting twist in engineering the electrical transport through magnetic skyrmions.

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Topological magnetic solitons such as magnetic skyrmions represent a class of particlelike magnetization textures that could serve as energy-efficient information bits of the future [1,2]. There are three important milestones which need to be reached in order to realize this vision, and which are currently an active field of research: the stabilization of room-temperature solitons [3,4], their deterministic control [5,6], and their deterministic readout [7,8]. With regard to the latter, noncollinear magnetic textures challenge us with broken translational invariance and variations on mesoscopic length scales. This is why the interpretation of experimental transport data is often strongly debated, as has been recently the case for $SrIrO_3/SrRuO_3$ bilayers [9,10]. While this system may exhibit skyrmionic magnetization textures, the presence of chiral domain walls and strong spin-orbit coupling (SOC) adds further complexity [11] and undermines an *a priori* gauge field interpretation of the observed topological Hall effect (THE) [12–14].

The THE has been used as a proxy for the detection of a skyrmion phase since the early days of skyrmionics [15], which is still actively pursued [16–19], and various theoretical approaches have been put forward in order to generalize upon its gauge-field interpretation. These are model based extensions of either the gauge-field language in the nonadiabatic regime [20,21], or on a T-matrix scattering theory operating in a weak-coupling regime [22,23]. The only approach that acknowledges the importance of SOC-induced gauge fields to linear order is that by Nakabayashi and Tatara [24]. However, previous work on the orbital magnetization of noncollinear spin textures suggests that any perturbative description in the SOC strength is eventually insufficient in a regime where no preferred spin reference frame exists anymore [25]. Strong SOC materials such as the above mentioned oxides can fall into this regime as well. Further, it would be desirable to have a formalism which can be generalized to an *ab initio* description and is not confined to the model level.

In this work, we close this gap and set the foundation for future progress in the area of electrical transport properties of isolated topological solitons. We do so by rephrasing its constitutive equations in the language of noncommutative geometry [26,27]. This procedure results in a systematic recipe to incorporate noncollinear magnetism by providing corrections to the conductivity tensor order by order in the gradients of the local magnetization texture. In addition, the transverse components of the conductivity tensor receive a natural interpretation as nontrivial elements in the cyclic cohomology class of the noncommutative phase space geometry [28]. We focus on the first nontrivial transverse correction in this expansion which we coin chiral Hall effect (CHE). We argue that the CHE provides an important contribution to the Hall effect in noncollinear magnets, which appear in addition to the topological Hall effect.

Central to our approach is the phase space formulation of quantum mechanics [29,30] and its application to the nonequilibrium Keldysh formalism [26]. In this approach, the noncommutative algebra of quantum operators is translated into the noncommutative $\star$ product

$$\star \equiv \exp\left[\frac{i\hbar}{2}(\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{p_\mu} - \overleftarrow{\partial}_{p_\mu}\overrightarrow{\partial}_{x^\mu})\right], \tag{1}$$

acting on phase space functions via left- and right-acting derivatives. Here, and in the following discussion, we refer to the four-position $(x)^\mu \equiv (ct, \mathbf{x})$ and momentum $(p)^\mu \equiv (\epsilon/c, \mathbf{p})$ with respect to the metric signature $(-+++)$. Both x and p can now be regarded as classical, commutative numbers. When H denotes the Hamiltonian and $\underline{\Sigma}$ the self-energy on phase space, the Dyson equation reads

$$(\epsilon - H - \underline{\Sigma})\star\underline{G} = \mathrm{id}, \tag{2}$$

whose solutions $\underline{G}$ represent the nonequilibrium Keldysh Green's function and encode the physical properties of the system. By the nature of the $\star$ product, a solution of the Dyson equation yields a semiclassical expansion with respect to $\hbar$ with the classical propagator at $\mathcal{O}(\hbar^0)$ represented by $\underline{G}_0 = (\epsilon - H - \underline{\Sigma})^{-1}$. In the following, we approximate the self-energy $\underline{\Sigma}$ with a constant broadening Γ in the advanced and retarded component $\Sigma^A = (\Sigma^R)^* = i\Gamma$ and the lesser component $\Sigma^< = 2if(\epsilon)\Gamma$. In the presence of static and homogeneous external electromagnetic fields, the $\star$ product is modified and takes the form [26]

$$\star \equiv \exp\left[\frac{i\hbar}{2}(\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{p_\mu} - \overleftarrow{\partial}_{p_\mu}\overrightarrow{\partial}_{x^\mu} + qF^{\mu\nu}\overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{p^\nu})\right]. \tag{3}$$

This equation introduces the electromagnetic field tensor $F_{\mu\nu}$ which follows the usual conventions. In particular, $cF_{0i} = E_i$, where c is the speed of light, $q = -e$ denotes the electric charge, and E_i represents the components of an applied electric field. Within the semiclassical formalism, the electric four-current density $j^\mu \equiv (c\rho, \mathbf{j})$ is given by

$$j^\mu \equiv \frac{e}{2}\Im\int\frac{dp}{(2\pi)^{d+1}\hbar^d}\mathrm{tr}\left\{\frac{\partial}{\partial p_\mu}(G^R)^{-1\star} \overset{\star}{,} G^<\right\}, \tag{4}$$

with $dp = d\epsilon d\mathbf{p}/c$, where $\{\overset{\star}{,}\}$ indicates the anticommutator with respect to the $\star$ product ($\Im$ is the imaginary part). The current fulfills the continuity equation $\partial_\mu j^\mu = 0$ (see the Supplemental Material [31]). We define the net current density as the average $\langle\mathbf{j}\rangle \equiv V^{-1}T^{-1}\int dx\,\mathbf{j}(\mathbf{x})$, where $dx = cdtd\mathbf{x}$, T represents the time interval of measurement, and V the d-dimensional volume of the sample. The $\star$ exponent featuring in Eq. (4) denotes the inverse with respect to the $\star$ product. The conductivity tensor is then defined as the derivative $\sigma_{kl} = \partial\langle j_k\rangle/\partial E_l|_{\mathbf{E}\to 0}$. In order to determine this derivative, we expand $\underline{G} \to \underline{G} + \underline{G}_{\mathbf{E}}$ to find the first-order correction due to the electric field (see Supplemental Material [31]):

$$\underline{G}_{\mathbf{E}} = \frac{i\hbar q}{2}(\underline{G}\star\nabla_{\mathbf{p}}\underline{G}_0^{-1}\star\partial_\epsilon\underline{G} - \underline{G}\star\partial_\epsilon\underline{G}_0^{-1}\star\nabla_{\mathbf{p}}\underline{G})\cdot\mathbf{E}, \tag{5}$$

where the $\star$ product and the Green's functions are now evaluated at zero field. The conductivity can then be split into two contributions $\sigma_{kl} = \sigma_{kl}^{\mathrm{sea}} + \sigma_{kl}^{\mathrm{surf}}$. Employing the convention

$$\mathfrak{Tr}(\bullet) \equiv \int\frac{dp}{(2\pi)^{d+1}\hbar^d}\mathrm{tr}(\bullet), \tag{6}$$

we arrive at the deformed Kubo-Bastin formula,

$$\sigma_{kl}^{\mathrm{sea}} = \hbar e^2\Re\mathfrak{Tr}_{\mathrm{sea}}\langle G^R\star v_k\star G^R\star v_l\star G^R - (k\leftrightarrow l)\rangle, \tag{7}$$

$$\sigma_{kl}^{\mathrm{surf}} = \hbar e^2\Re\mathfrak{Tr}_{\mathrm{surf}}\langle v_k\star(G^R - G^{\mathrm{A}})\star v_l\star G^{\mathrm{A}}\rangle. \tag{8}$$

Here, $\Re$ is the real part and we define $\mathfrak{Tr}_{\mathrm{sea}}(\bullet) = \mathfrak{Tr}[f(\epsilon)\bullet]$ and $\mathfrak{Tr}_{\mathrm{surf}}(\bullet) = \mathfrak{Tr}[f'(\epsilon)\bullet]$, where f is the Fermi distribution. The derivation of these equations heavily relies on certain algebraic properties of the $\star$ product. In particular its associativity and the cyclic property of the phase space trace, $\mathfrak{Tr}\langle A\star B\rangle = \mathfrak{Tr}\langle B\star A\rangle$, which would not be valid without the real-space averaging.

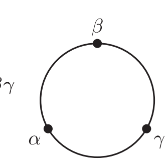
We now want to comment on the geometrical interpretation of the anomalous Hall effect for the special case $d = 2$ of two space dimensions. For this, we introduce the objects $\theta_\mu \equiv [\underline{G}\star(\partial/\partial p^\mu)(\underline{G})^{-1\star}]$ [32], and by defining

$$\varphi^2(\theta_0, \theta_1, \theta_2) \equiv \Re\mathfrak{Tr}_{\mathrm{sea}}\langle\epsilon^{\mu_0\mu_1\mu_2}\theta_{\mu_0}^R\star\theta_{\mu_1}^R\star\theta_{\mu_2}^R\rangle, \tag{9}$$

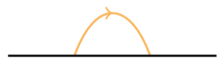
where $\mu \in \{\epsilon, p_x, p_y\}$, and ϵ is the fully antisymmetric tensor, we can write the Hall conductivity as

$$\sigma_{xy}^{\mathrm{sea}} = \frac{\hbar e^2}{3}\Re\varphi^2(\theta_0, \theta_1, \theta_2). \tag{10}$$

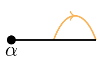
In the language of noncommutative geometry [27], φ^2 represents a cyclic cocycle, i.e., it represents the cyclic cohomology class of the electronic system (this is shown explicitly in the Supplemental Material [31]; see also Ref. [33] for an introduction to the subject). It can be seen as a generalization of the Thouless-Kohmoto-Nightingale-Nijs (TKNN) formula. In fact, for insulators in the semiclassical limit $\hbar \to 0$, one retains the form of the generalized TKNN formula $\sigma_{xy}^{\mathrm{sea}} = (e^2/h)N_2$ [34,35], where N_2 is the Chern character of the electronic system. Also in the general case, a smooth variation of the Green's function δG^R will not alter φ^2 for insulators at zero temperature, i.e., $\delta\varphi^2 = 0$. A proof of this statement can be obtained by generalizing an argument of Ref. [36] (see Supplemental Material [31]). That it should be quantized can be understood from the considerations in noncommutative Chern-Simons theory [37]. Symbolically, we will write the Kubo 2-cocycle as

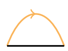
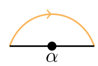
$$\varphi^2(\theta_0, \theta_1, \theta_2) = \Re\,\mathfrak{Tr}_{\mathrm{sea}}\,\langle\epsilon^{\alpha\beta\gamma}$$  $$\rangle, \tag{11}$$

which puts an emphasis on the cyclic invariance of the trace operation and on its geometrical origin. Here, the solid lines represent retarded Green's functions while the vertices represent derivatives of the inverse propagator, i.e., $\underset{\alpha}{\bullet} \equiv \partial_{p^\alpha}(G_0^R)^{-1}$. In order to translate the diagram into an equation it is to be read in the clockwise direction. The next-order correction (the "one-loop" level) can be obtained by expanding the Green's functions to first order in the gradients [26] $G^R = G_0^R + (i\hbar/2)\Pi^{ij} G_0^R \partial_i (G_0^R)^{-1} G_0^R \partial_j (G_0^R)^{-1} G_0^R$, where the tensor Π encodes the first-order expansion of the $\star$ product. For time-independent Hamiltonians, the indices i and j therefore run over the components of $\mathbf{x}$ and $\mathbf{p}$. Diagrammatically, this correction can be expressed as

$$G^{\mathrm{R}} = G_0^{\mathrm{R}} +$$  . (12)

Inserting this first-order expansion into Eq. (11), expanding the $\star$ products and retaining only first-order contributions in $\hbar$ leads to the full set of diagrams presented in the Supplemental Material [31]. All corrections can be incorporated into a single renormalized (dressed) four-momentum vertex $\bar{\alpha}$, concisely written as

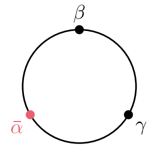
$\underset{\bar{\alpha}}{\bullet} = 3\Big($ 

$- \partial_{p_\alpha}$  $\Big) - 2$ 

. (13)

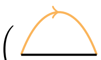
Going from the zeroth order to the one-loop corrections then amounts to the replacement $\underset{\alpha}{\bullet} \to \underset{\bar{\alpha}}{\bullet}$ in Eq. (11). Since the renormalized vertex is by construction linear in the gradients of the magnetization texture, we refer to the resulting first order approximation to $\sigma_{xy}^{\mathrm{sea}}$ as the chiral Hall effect $\sigma_{xy}^{\mathrm{che}}$ (CHE). While the magnetization has even parity, its derivatives will change sign under inversion. The local Hamiltonian thus needs a broken inversion symmetry to reveal a finite CHE which is in a qualitative difference to the THE which can arise even in skyrmions favored by frustrated interactions in inversion symmetric crystals. The latter has also a distinctly different scaling behavior compared to the CHE as it depends on a different number of gradients (an example can be found in the Supplemental Material [31]).

Consider a generic time-independent two-band Hamiltonian of the form $H \equiv d_\mu \tau^\mu$, where $\mu \in \{0, 1, 2, 3\}$, $\tau^\mu = (\mathrm{id}, \boldsymbol{\tau})$ and $d^\mu = (d^0, \mathbf{d})$ in the $(+ - - -)$ metric convention. $\boldsymbol{\tau}$ denotes the vector of Pauli matrices. We consider H as the phase space version of some quantum Hamiltonian and therefore, d^μ is an arbitrary function of the phase space coordinates $\mathbf{x}$ and $\mathbf{p}$. Defining the four-vector $g^\mu = (g^0, \mathbf{g}) \equiv (\epsilon - d^0 + i\Gamma, \mathbf{d})$, the retarded Green's function for this Hamiltonian is given by $G_0^R = g_\mu \tau^\mu / g^\nu g_\nu$. With these conventions, the first order gradient correction can be obtained via a suitable renormalization of the current vertex as described above. The renormalized diagram evaluates to

$\mathrm{tr}\ \epsilon^{\alpha\beta\gamma}$ 

$$= \frac{-4i}{(g^\mu g_\mu)^2} \Big(\mathbf{d} \cdot (\partial_{p_x} \mathbf{d} \times \bar{\mathbf{v}}_{p_y}) + \mathbf{d} \cdot (\bar{\mathbf{v}}_{p_x} \times \partial_{p_y} \mathbf{d}) + g^0\, \bar{\mathbf{v}}_\epsilon \cdot (\partial_{p_y} \mathbf{d} \times \partial_{p_x} \mathbf{d}) + \bar{v}_\epsilon^0\, \mathbf{d} \cdot (\partial_{p_x} \mathbf{d} \times \partial_{p_y} \mathbf{d}) \Big), \tag{14}$$

where we have defined $\bar{v}_\alpha^\mu \equiv \mathrm{tr}\underset{\alpha}{\bullet}\tau^\mu/2$. Recalling that the Berry curvature of a two-level system is given by $\Omega^{ij} = -\mathbf{d} \cdot (\partial_i \mathbf{d} \times \partial_j \mathbf{d})/(2|\mathbf{d}|^3)$, one can interpret the terms in this equation as originating from its first order correction. For example, by evaluating the coupling to $\bar{v}_\epsilon^0$, one finds

$\bar{v}_\epsilon^0 = -\frac{1}{2}\partial_\epsilon \mathrm{tr}\Big($  $\Big) = \partial_\epsilon \frac{2|\mathbf{d}|^3}{g_\mu g^\mu} \sum_i \Omega^{x_i p_i}.$ (15)

The term proportional to $\bar{v}_\epsilon^0$ thus describes a coupling of the momentum space Berry curvature $\Omega^{p_x p_y}$ to the mixed space Berry curvatures $\Omega^{x_i p_i}$—a connection which was speculated on before (but not shown to exist) for the chiral orbital magnetization of isolated skyrmions [25]. The appearance of a term like this is remarkable, since the semiclassical dynamics of insulators would be well captured by the Středa formula with a modified density of states given by the Pfaffian $\mathcal{D} = |\mathrm{Pf}(\boldsymbol{\Omega} - \boldsymbol{\Pi})|/(2\pi)^2$ [38–40] and while a coupling like $\Omega^{p_x p_y}\Omega^{x_i p_i}$ is not present in the two-dimensional expansion of the Pfaffian, it can emerge from adiabatic pumping in inhomogenous crystals [41,42].

Finally, we demonstrate the existence of the CHE by performing an explicit numerical simulation for a spin-polarized electronic system with spin-orbit interaction. Namely, we focus our analysis on the two-dimensional magnetic Rashba model, which is well suited to describe the effect of SOC in interfacial systems:

$$H = \frac{\mathbf{p}^2}{2m_{\mathrm{eff}}^*} + \alpha_R(\boldsymbol{\tau} \times \mathbf{p})_z + \Delta_{\mathrm{xc}} \boldsymbol{\tau} \cdot \hat{\mathbf{n}}(\mathbf{x}), \tag{16}$$

where m_{eff}^* is the electron's (effective) mass, α_R is the Rashba spin-orbit coupling constant, and Δ_{xc} is the strength of the local exchange field which mediates the coupling to the local magnetization texture parameterized by $\hat{\mathbf{n}}(\mathbf{x})$. This Hamiltonian can therefore be cast into the generic form $H \equiv d_\mu \sigma^\mu$ from the previous section with $d^0 = \mathbf{p}^2/(2m_{\mathrm{eff}}^*)$ and $\mathbf{d} = \alpha_R \mathbf{e}_z \times \mathbf{p} - \Delta_{\mathrm{xc}} \hat{\mathbf{n}}$. In general, the CHE can be formulated in a tensorial way as

$$\sigma_{xy}^{\mathrm{che}}[\hat{\mathbf{n}}] = \frac{1}{V} \int d\mathbf{x} (\sigma_{xy}^{\mathrm{che}})^{ij}(\hat{\mathbf{n}}) \partial_i \hat{n}_j, \tag{17}$$

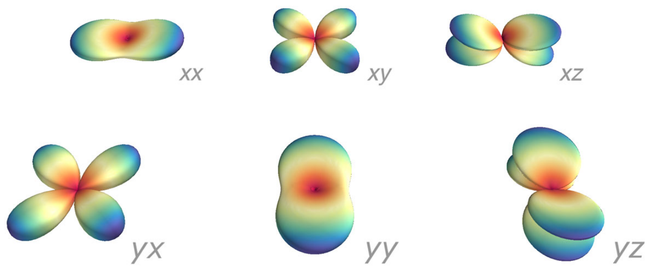


FIG. 1. Angular dependence of the chiral Hall effect in the Rashba model. The shown matrix represents the normalized angular magnetization dependence of the components of the CHE which couple to the real-space derivatives $\partial_i \hat{n}_j$, where $i \in \{x, y\}$ is the row index and $j \in \{x, y, z\}$ is the column index of the figure ($\hbar\alpha_R = 1.41$ eV Å, $\Delta_{xc} = 0.5$ eV, $\mu = 0$, $\Gamma = 50$ meV, $m^*_{\text{eff}} = 3.81 m_e$). All shapes have an inversion center resulting from Onsager reciprocity.

where $(\sigma^{\text{che}}_{xy})^{ij}(\hat{\mathbf{n}})$ depends only on the direction of $\hat{\mathbf{n}}$, but not on its derivatives. In Fig. 1, we present the zero-temperature directional dependence of this tensor for the case of $\hbar\alpha_R = 1.41$ eV Å, $\Delta_{xc} = 0.5$ eV, $\Gamma = 50$ meV, and $m^*_{\text{eff}} = 3.81 m_e$. The chemical potential was set to $\mu = 0$ eV. All shapes have an inversion center due to the local Onsager reciprocity, which dictates $(\sigma^{\text{che}}_{xy})^{ij}(-\hat{\mathbf{n}}) = (\sigma^{\text{che}}_{xy})^{ij}(\hat{\mathbf{n}})$ as the correct behavior under time reversal and which is proved in the Supplemental Material [31] (generalizing results from Ref. [43]). Notably, the angular dependence does not resemble a dominant n_z^2 behavior in $(\sigma^{\text{che}}_{xy})^{xx}(\hat{\mathbf{n}})$ and $(\sigma^{\text{che}}_{xy})^{yy}(\hat{\mathbf{n}})$ as would be expected for the Berry phase term which was alluded to above. This means that the Berry curvature contribution to the CHE is subdominant in this case, and it should therefore be rather interpreted as a strongly nonadiabatic effect where an interpretation in terms of emergent magnetic fields is not applicable anymore. The SOC induced anisotropy of the CHE tensor is a central ingredient for the observation of a large effect. One can also imagine that a local frustrated spin arrangement resulting in a finite scalar spin chirality can replace SOC in triggering the necessary symmetry breaking.

It lies in the power of our approach that once the directional dependence of the CHE tensor is known, it can be integrated for arbitrary magnetization textures. We chose to perform a numerical experiment using the spin dynamics code *Spirit* [44], and perform a hysteresis loop experiment for a Heisenberg magnet with ferromagnetic nearest neighbor exchange $J = 1$ meV, Néel type Dzyaloshinskii-Moriya interaction (DMI) $D = 0.5$ meV and spin moment $\mu_s = 2\mu_B$ at a temperature of $T = 1$ K. This system stabilizes Néel skyrmions in between 2 and 3 T (for further details we refer to the Supplemental Material [31]) [45]. This kind of hysteresis experiment is commonly conducted in materials where skyrmions are so small that optical techniques are not available for their detection. Instead, the presence of skyrmions is inferred from an additional feature in the anomalous Hall signal which is commonly attributed to the topological Hall effect.

The results of this numerical experiment, presented in Fig. 2(a), demonstrate the emergence of the chiral Hall effect in the switching process and its correlation with the buildup of the topological charge. However, unlike the topological Hall effect it is not caused by this charge. This becomes evident when we look closer at the field strength of $H = 0.57$ T, at which local conical phases appear in the sample. Remarkably, the CHE shows a very large magnitude in the areas that exhibit conical spin spirals [see the inset of Fig. 2(b) of the main text, and Fig. 3 in the Supplemental Material [31]), while the topological charge is still zero in this case. Within the field regime where magnetic skyrmions are stabilized, the CHE shows a nontrivial behavior, and manifests itself in ringlike textures in the local current density, see the inset of Fig. 2(c).

Overall, our findings call for a reinterpretation of common transport experiments in chiral magnets. In the

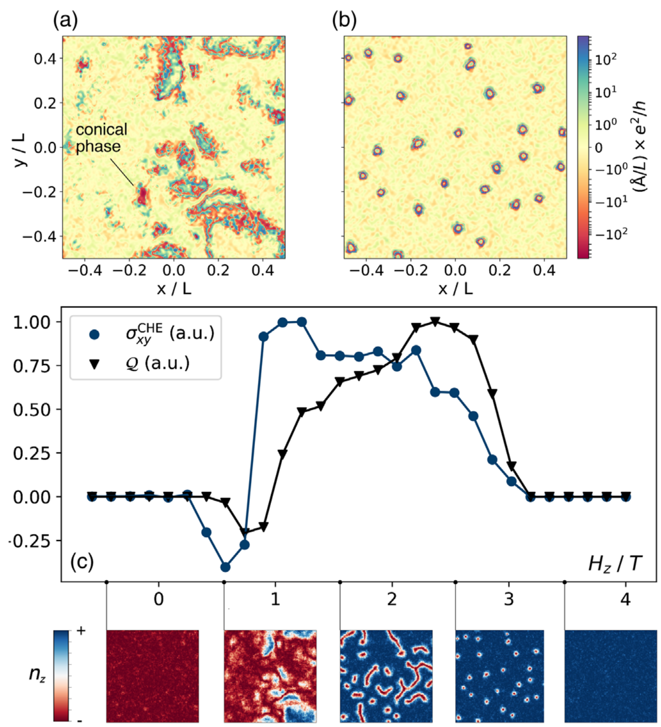


FIG. 2. Chiral Hall effect in a chiral Heisenberg magnet. A 2D system of side-length L containing 100×100 spins is simulated at $T = 1$ K using the *Spirit* code. Starting from the polarized state at $H_z = -4$T, the magnetic field is increased step by step. (a) At $H_z = 0.57$ T the magnetization starts to switch. Shown is the density of the CHE which displays strong resonances at local conical phases which have no topological charge. (b) At $H_z = 2.53$ T, the system is dominated by magnetic skyrmions which exhibit a ring-like CHE signature. (c) Tracing the evolution of the average CHE and the topological charge $\mathcal{Q}$ one finds a correlated signal (the corresponding magnetic state is shown in the inset below). Yet, by its nature, the CHE is not caused by $\mathcal{Q}$.

past, these mostly relied on phenomenological arguments based on the adiabatic theory of the topological Hall effect, which is already known to be insufficient in many cases [20–23]. Our formalism represents the first systematic derivation of corrections to the anomalous Hall effect of collinear magnets, which are in principle not restricted to the model level or the weak-coupling regime, and which can be certainly reformulated in the language of *ab initio* electonic structure.

We believe that the CHE will provide an essential degree of freedom in engineering the electrical transport through magnetic skyrmions and noncollinear magnetic textures in general. This is emphasized by the fact that according to our analysis the nonvanishing CHE emerges already for one-dimensional spin textures such as spin spirals and domain walls. In materials with pronounced spin-orbit interaction the CHE can be a significant contribution to the overall Hall signal. This might be the case, for example, in the aforementioned $SrIrO_3/SrRuO_3$ bilayers or the celebrated B20-compound MnSi, where scattering off the cone phase has been speculated to lead to a Hall effect before [46] as well as in metal-insulator heterostructures which display THE-like features [17]. But also in antiferromagnetic systems of, e.g., the Mn_3X family, which exhibit the "topological" Hall effect, an investigation of the CHE can be of great interest [47,48].

Importantly, we have embedded the approach to the CHE into the rich context of noncommutative geometry which lends itself to a deep interpretation. This connection is fruitful for two reasons. First, its language might shed some light on how superlattices of magnetic skyrmions can induce topological states in magnetic insulators as they change the topology of the underlying noncommutative phase space. Second, electronic transport in noncollinear magnets on its own represents an experimentally well-established field, where theoretical ideas from noncommutative geometry and noncommutative Chern-Simons theory could be put under scrutiny, while they remain rather elusive in the context of high-energy physics where they have been studied intensively 20 years ago [49–51].

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[1] A. Fert, N. Reyren, and V. Cros, Nat. Rev. Mater. **2**, 17031 (2017).
[2] W. Kang, Y. Huang, X. Zhang, Y. Zhou, and W. Zhao, Proc. IEEE **104**, 2040 (2016).
[3] S. Woo, K. Litzius, B. Krüger, M.-Y. Im, L. Caretta, K. Richter, M. Mann, A. Krone, R. M. Reeve, M. Weigand, P. Agrawal, I. Lemesh, M.-A. Mawass, P. Fischer, M. Kläui, and G. S. D. Beach, Nat. Mater. **15**, 501 (2016).
[4] W. Legrand, D. Maccariello, F. Ajejas, S. Collin, A. Vecchiola, K. Bouzehouane, N. Reyren, V. Cros, and A. Fert, Nat. Mater. **19**, 34 (2019).
[5] K. Litzius, I. Lemesh, B. Krüger, P. Bassirian, L. Caretta, K. Richter, F. Büttner, K. Sato, O. A. Tretiakov, J. Förster *et al.*, Nat. Phys. **13**, 170 (2017).
[6] W. Jiang, X. Zhang, G. Yu, W. Zhang, X. Wang, M. B. Jungfleisch, J. E. Pearson, X. Cheng, O. Heinonen, K. L. Wang *et al.*, Nat. Phys. **13**, 162 (2017).
[7] K. Hamamoto, M. Ezawa, and N. Nagaosa, Appl. Phys. Lett. **108**, 112401 (2016).
[8] D. Maccariello, W. Legrand, N. Reyren, K. Garcia, K. Bouzehouane, S. Collin, V. Cros, and A. Fert, Nat. Nanotechnol. **13**, 233 (2018).
[9] J. Matsuno, N. Ogawa, K. Yasuda, F. Kagawa, W. Koshibae, N. Nagaosa, Y. Tokura, and M. Kawasaki, Sci. Adv. **2**, e1600304 (2016).
[10] Y. Ohuchi, J. Matsuno, N. Ogawa, Y. Kozuka, M. Uchida, Y. Tokura, and M. Kawasaki, Nat. Commun. **9**, 213 (2018).
[11] K.-Y. Meng, A. S. Ahmed, M. Baćani, A.-O. Mandru, X. Zhao, N. Bagués, B. D. Esser, J. Flores, D. W. McComb, H. J. Hug *et al.*, Nano Lett. **19**, 3169 (2019).
[12] P. Bruno, V. K. Dugaev, and M. Taillefumier, Phys. Rev. Lett. **93**, 096806 (2004).
[13] K. Y. Bliokh and Y. P. Bliokh, Ann. Phys. (Amsterdam) **319**, 13 (2005).
[14] C. Franz, F. Freimuth, A. Bauer, R. Ritz, C. Schnarr, C. Duvinage, T. Adams, S. Blügel, A. Rosch, Y. Mokrousov *et al.*, Phys. Rev. Lett. **112**, 186601 (2014).
[15] A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni, Phys. Rev. Lett. **102**, 186602 (2009).
[16] K. Zeissler, S. Finizio, K. Shahbazi, J. Massey, F. Al Ma'Mari, D. M. Bracher, A. Kleibert, M. C. Rosamond, E. H. Linfield, T. A. Moore *et al.*, Nat. Nanotechnol. **13**, 1161 (2018).
[17] Q. Shao, Y. Liu, G. Yu, S. K. Kim, X. Che, C. Tang, Q. L. He, Y. Tserkovnyak, J. Shi, and K. L. Wang, National electronics review **2**, 182 (2019).
[18] M. Raju, A. Yagil, A. Soumyanarayanan, A. K. Tan, A. Almoalem, F. Ma, O. Auslaender, and C. Panagopoulos, Nat. Commun. **10**, 696 (2019).
[19] A. S. Ahmed, A. J. Lee, N. Bagués, B. A. McCullian, A. M. A. Thabt, A. Perrine, P.-K. Wu, J. R. Rowland, M. Randeria, P. C. Hammel, D. W. McComb, and F. Yang, Nano Lett. **19**, 5683 (2019).
[20] K. Nakazawa, M. Bibes, and H. Kohno, J. Phys. Soc. Jpn. **87**, 033705 (2018).
[21] L. Vistoli, W. Wang, A. Sander, Q. Zhu, B. Casals, R. Cichelero, A. Barthélémy, S. Fusil, G. Herranz, S. Valencia *et al.*, Nat. Phys. **15**, 67 (2019).
[22] K. S. Denisov, I. V. Rozhansky, N. S. Averkiev, and E. Lähderanta, Phys. Rev. B **98**, 195439 (2018).

[23] I. Rozhansky, K. Denisov, M. Lifshits, N. Averkiev, and E. Lähderanta, Phys. Status Solidi B **256**, 1900033 (2019).
[24] N. Nakabayashi and G. Tatara, New J. Phys. **16**, 015016 (2014).
[25] F. R. Lux, F. Freimuth, S. Blügel, and Y. Mokrousov, Commun. Phys. **1**, 60 (2018).
[26] S. Onoda, N. Sugimoto, and N. Nagaosa, Prog. Theor. Phys. **116**, 61 (2006).
[27] A. Connes and S. Berberian, *Noncommutative Geometry* (Elsevier Science, San Diego, 1994).
[28] J. Bellissard, A. van Elst, and H. Schulz-Baldes, J. Math. Phys. (N.Y.) **35**, 5373 (1994).
[29] J. E. Moyal, in *Math. Proc. Camb. Philos. Soc.* (Cambridge University Press, Cambridge, England, 1949), Vol. 45, pp. 99–124.
[30] H. J. Groenewold, in *On the Principles of Elementary Quantum Mechanics* (Springer, Dordrecht, 1946), pp. 1–56.
[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.096602 for details on the mathematical derivations and the spin dynamics simulation.
[32] These can be interpreted as the components of a noncommutative version of the Maurer-Cartan form.
[33] M. Khalkhali, arXiv:math/0408416.
[34] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[35] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[36] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[37] M. Sheikh-Jabbari, Phys. Lett. B **510**, 247 (2001).
[38] P. Streda, J. Phys. C **15**, L717 (1982).
[39] D. Xiao, J. Shi, and Q. Niu, Phys. Rev. Lett. **95**, 137204 (2005).
[40] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
[41] D. Xiao, J. Shi, D. P. Clougherty, and Q. Niu, Phys. Rev. Lett. **102**, 087602 (2009).
[42] J.-H. Zhou, H. Jiang, Q. Niu, and J.-R. Shi, Chin. Phys. Lett. **30**, 027101 (2013).
[43] L. Banyai and K. Elsayed, Ann. Phys. (N.Y.) **233**, 165 (1994).
[44] G. P. Müller, M. Hoffmann, C. Dißelkamp, D. Schürhoff, S. Mavros, M. Sallermann, N. S. Kiselev, H. Jónsson, and S. Blügel, Phys. Rev. B **99**, 224414 (2019).
[45] Equally well, one could study Bloch type DMI, but this would require a different kind of symmetry in the SOC term and would then lead to similar results.
[46] S. A. Meynell, M. N. Wilson, J. C. Loudon, A. Spitzig, F. N. Rybakov, M. B. Johnson, and T. L. Monchesky, Phys. Rev. B **90**, 224419 (2014).
[47] J. Kübler and C. Felser, Europhys. Lett. **108**, 67001 (2014).
[48] A. K. Nayak, J. E. Fischer, Y. Sun, B. Yan, J. Karel, A. C. Komarek, C. Shekhar, N. Kumar, W. Schnelle, J. Kübler *et al.*, Sci. Adv. **2**, e1501870 (2016).
[49] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
[50] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[51] R. J. Szabo, Phys. Rep. **378**, 207 (2003).