

**Integral Methods
for the Calculation
of Electric Fields**

Forschungszentrum Jülich GmbH
Scientific Series of the International Bureau

Integral Methods for the Calculation of Electric Fields

For Application in High Voltage Engineering

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German-Yugoslav-Cooperation
in Scientific Research and Technological Development

Herausgeber Forschungszentrum Jülich GmbH
und Vertrieb: ZENTRALBIBLIOTHEK
Postfach 1913 · D-5170 Jülich
Telefon (02461) 61-5368 · Telefax (02461) 61-6103

Druck: Weka-Druck GmbH, Linnich
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Scientific Series of the International Bureau, Volume 10

ISSN 0938-7676

ISBN 3-89336-084-0

Dieser Bericht ist auf umweltfreundlichem Papier gedruckt

The report is supported by the International Buerau of KFA Jülich - Germany.

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Chapter 1

Introduction

In addition to the laws of the physics of electric discharges, the knowledge of the electric field is the basis of the design of insulation systems in high voltage engineering. Although the properties of the insulation material and the environmental conditions are important, the electric field strength is ultimately the decisive parameter for the appearance of electric discharges. Particularly in the case of gas insulated substations, the predetermination of discharge voltages based on the knowledge of electric field distribution is an essential step during the design process.

In the past, experimental methods were used to determine the electric field, for example the electrolytic tank or semiconductive paper. Now these methods have been successfully replaced in most cases by numerical field calculation, and the recent developments in computer technology open new and interesting possibilities for the future. A variety of numerical methods and computer codes are available for both 2D and 3D electric fields. Whereas in the past the calculation of 3D electrostatic fields without symmetry was limited to main frame computers only, there are now possibilities to calculate these fields on graphic workstations and even on personal computers as well.

Many computer codes are available for the calculation of electric fields based on Finite Element Method (FEM). In the application of this method the whole space of interest must be subdivided into elements. One advantage of this discretization procedure is that the possibility of nonlinear properties of the space can be taken into consideration. On the other hand, this kind of discretization may result in a very high number of elements, especially in the case of 3D fields without any symmetry or in such cases where a very dense nodal distribution is necessary.

An alternative to the widely used FEM are so called integral meth-

ods like the Boundary Element Method (BEM) or the Charge Simulation Method (CSM). The basis for these methods is the integral formulation of Laplace's or Poisson's equation. It is a characteristic feature of these methods that generally the spatial discretization is limited only to the surface domain of electrodes or dielectric boundaries. This means that this kind of discretization can be done with a considerably smaller number of elements in comparison with FEM. On the other hand this significant saving of elements results in a full matrix of the equation system in comparison with the sparse matrix in the case of FEM, and the limitation of the discretization to the surface domain does not include nonlinearities in the space. But nevertheless these integral methods promise to be an economical way of numerical field calculation in high voltage engineering, particularly in the case of 3D fields without symmetries. A higher degree of accuracy can be reached with less computation time and core storage requirements. The above mentioned shortcoming of BEM and CSM with respect to the consideration of nonlinearities does not play an important role in high voltage engineering due to the fact that in most cases the dielectric properties of the insulation do not depend on the field strength distribution in fields without space charges. If there is a need to consider nonlinearities, modifications of BEM are available to handle this problem as well.

These possibilities of integral methods gave reason to initiate activities to improve these methods regarding the application in high voltage engineering and to develop appropriate computer codes on this basis. One of these activities is a cooperation between the Electromagnetic Field Group of the Faculty of Electrical Engineering, University of Sarajevo and the High Voltage Institute, Technical University of Munich. This cooperation started in 1987 and it is supported by the International Bureau of KFA Jülich. During this cooperation contacts were established to Prof. Dr. Peter Levin of Worcester Polytechnic Institute (USA) and to the project VENUS, a cooperation on field calculation in high voltage engineering between German universities (Hamburg, Karlsruhe, Munich) and the firm Asea Brown Boveri (ABB).

The practical result of the cooperation between Sarajevo and Munich was the development and the improvement of computer programs based on Boundary Element Method and Charge Simulation Method. The methods used for these codes and the programs themselves are described in this report.

Chapter 2

Appropriate Methods for Electric Field Calculation

One of the most important and most difficult stages in the process of designing of electromagnetic devices is the estimation of the electromagnetic processes that happen in it. The theoretical model of the electromagnetic process is based on the fundamental laws of electrodynamics stated in Maxwell's equations along with appropriate boundary conditions. In that way, from the mathematical perspective the task of estimating of the electromagnetic process is reduced to a boundary value problem described by a system of partial differential equations. The theoretical model, supplemented with the information about the geometrical characteristics of the system, electromagnetic characteristics of the media where the processes happen, and the characteristics of the sources of the field, contain all the information about electromagnetic processes of interest in the examined electromagnetic device (eg. field lines, equipotential lines, location of the maximal electrical stress, electromagnetic forces, inductivity, etc.).

The aim of determining the electromagnetic process is to obtain from the theoretical model the information about the electromagnetic process which is necessary and appropriate for the analysis of the field structures in the examined electromagnetic device, in qualitative or quantitative form, using an appropriate mathematical model. Introducing certain assumptions and simplifications, and neglecting some mutual coupling influences in the theoretical model, the mathematical model of the electromagnetic process in a specific electromagnetic device can often be obtained. Thus, considering boundary conditions, certain mathematical transformations are performed on the theoretical model. Depending on the sort of problem some basic

functions of the field could be introduced (for example scalar function of the electric or magnetic field potential, function of the vector potential of the magnetic field, etc). Consequently, there can be various forms of the mathematical model for a specific electromagnetic process.

Depending on the form of the unknown field function in the governing equations, mathematical models can be conditionally divided as follows:

- differential (the mathematical model is described by a system of the partial differential equations with appropriate boundary conditions),
- integral (the mathematical model has a form of an integral equation system, with boundary conditions implicitly prescribed in the equations system),
- differential - integral (the combination of the previously given models, whereby the electromagnetic process in some parts is described by differential equations, and in other by integral equations).

Solving the mathematical models using analytical methods is very limited and only occasionally applicable, mostly on simple geometrical systems, simple boundary conditions, and with constant electromagnetic characteristics. The development of fast digital computers has put into the foreground the use of the numerical methods when solving mathematical models of the electromagnetic processes.

When solving the differential mathematical models, two numerical approaches are used:

- Finite Difference Method (FDM),
- Finite Element Method (FEM).

The Finite Difference Method has been used for a long time in the practice for the calculation of the two-dimensional (2D) and axi-symmetric electrostatic and magnetostatic fields. On the basis of this method, it is possible to translate a set of partial differential equations into a set of difference equations, using an appropriately chosen orthogonal mesh. Once the points of the mesh inside the area of interest are numbered, the system of difference equations is reduced to a system of linear algebraic equations of a very high order. The basic characteristic of this system is a very sparse coefficient matrix. This method gives relatively good results in the calculation of the 2D models of high-voltage apparatus and electric machines when taking into consideration the nonlinear magnetization curve. The drawbacks of this method are difficulties in setting up of the mesh which well approximates the boundary lines between different media, the large number of required input data, and a need for numerical differentiation.

The Finite Element Method is successfully used in solving 2D and 3D electromagnetic fields, and for other technical applications. This method is based on the variational calculation, where the solution of field problems is reduced to obtaining the stationary state of the appropriate energy functional. Starting with this fact, and taking into consideration that among enough small parts of the observed area (finite elements), the field function can be expressed as an analytical function of the position within the finite element, we arrive to a system of algebraic equations. Solving it, it is possible to derive the information about the field function in the area of interest. The result here is also a matrix equation of relatively high order, with a very sparse coefficient matrix, which can be solved using iterative methods or some special direct methods. The drawback of this method is a generally large number of input data that are necessary when defining the geometrical and material parameters of the selected finite elements of the area.

For the numerical solution of the integral models, the Charge Simulation Method and the Boundary Element Method are used.

In the Charge Simulation Method, equivalent charges (point, line, ring charges) placed in a vacuum are used. The values of these charges are determined in such a way that the boundary conditions are satisfied. The position of those charges is selected so that the singularity of the integral operator is avoided, and that the coefficients of the algebraic matrix equation can be analytically determined. This method has found its application in the analysis of the electrostatic fields in high voltage engineering. The method was extended by the introduction of equivalent area charges placed directly on the surface of electrodes and dielectric boundaries.

In the Boundary Element Method, integral equations are translated into a system of algebraic equations replacing the surface and volume integral by the sums of the well selected small curvilinear surfaces over the boundary between two media with different characteristics, or, of small volumes inside of the media with non-linear electromagnetic characteristics (Surface Boundary Elements - SBE and Volume "Boundary" Elements - VBE). Over the boundary element, the field is expressed as an analytical function evaluated at certain interpolation nodes. The characteristics of the media are usually taken as constant values for a certain element. The matrix of coefficients of algebraic equations, derived in this way, is of relatively low order. However, it is a full matrix with non-constant coefficient values in case of non-linear media. For the solution of this type of matrix algebraic equation, most frequently, a combination of direct matrix inversion and an iterative procedure for non-linear parameter adjustment along the boundary elements, is used. The weak side of this method is the full coefficient matrix and the relatively large consumption of the computation time for the

calculation of the matrix coefficient (numerical calculation of an integral). Advantages of this method are the relatively small number of input data necessary for the exact description of all the characteristics of the system, and placing of the boundary elements only over the boundaries between the different media. This method gives distinctly high accuracy, because the information about boundary conditions is directly presented in a system of algebraic equations, and it also makes possible the solution of so called "unbounded fields" where the application of the FDM and FEM is more difficult.

A modification of the Boundary Element Method, based on the introduction of the Galerkin formulation, is presented in [21]. Here, instead of requiring that the error function be zero at discrete points, the condition that the residual (error function), multiplied by a weighting function, integrates to zero over the whole domain is imposed.

In a great number of practical electromagnetic devices, the electromagnetic process is stationary, or it could be viewed as approximately stationary (quasi-stationary fields). Therefore, mathematical model of the electromagnetic field is divided in two mathematical models:

- mathematical model of the electrostatic field,
- mathematical model of the magnetostatic field.

In most devices of this kind, the information about one of two fields only is important for the design and analysis of the electromagnetic device. Therefore, from this aspect electromagnetic devices could be divided into electrical apparatus and magnetic apparatus.

Electrical apparatus are most frequently without "primary" field sources, geometrically complex, with several different media and constant electric parameters. The most important information in the analysis of these apparatus are the maximum electrical stress and the location where this stress appears. The reduction of electrical stress and optimization of geometrical dimensions of the electrical device are the basic criteria in the process of the design of these devices, so the numerical calculations are performed in such way as to yield the above mentioned information. For the numerical calculation of electric field, all above mentioned numerical methods are practically used.

Chapter 3

The Mathematical Methods and Numerical Procedures

3.1 Charge Simulation Method

3.1.1 Principle

Charge Simulation Method (CSM) is based on the principle that the surface charges on the electrodes or dielectric interfaces are replaced by a set of simulation charges located outside the field domain or on the boundary. The position and the type of these simulation charges are predetermined, the magnitude is unknown. These magnitudes of the simulation charge have to be calculated so that their integrated effect satisfies the boundary conditions exactly at a selected number of collocation points. The superposition of the potentials of the simulation charges must be equal to the boundary potential Φ_B of the collocation point B :

$$\Phi = \sum_{j=1}^n p_j \cdot Q_j \quad (3.1)$$

where Q_j is the magnitude of the j -th charge and p_j is its assigned potential coefficient. Usually the number of simulation charges and collocation points is the same, then the application of this equation to all collocation points leads to a system of n linear equations for the n charges:

$$[P][Q] = [\Phi_B] \quad (3.2)$$

The elements of the $(n \times n)$ matrix $[P]$ depend on the type and the position of the charge. The vector $[\Phi_B]$ contains the potential of the collocation points on the boundaries. After the solution of the equation system for the unknown magnitudes of the charges one must check whether the calculated simulation charges fit the boundary conditions. If this check is sufficient then the potential Φ and the field strength E in any point $A(x, y, z)$ in the field domain can be calculated by means of analytical expressions:

$$\Phi(x, y, z) = \sum_{j=1}^n p_j(x, y, z) \cdot Q_j \quad (3.3)$$

$$\vec{E}(x, y, z) = - \sum_{j=1}^n [\nabla p_j(x, y, z)] \cdot Q_j \quad (3.4)$$

The name "Charge Simulation Method" usually implies the idea of the substitution of the real charges by a set of point, line and ring charges ("discrete charges"). However, in principle procedures where area charges, volume charges or multipoles are used for the substitution also belong to the family of Charge Simulation Methods. Particularly the substitution by area charges has proven to be a valuable extension of CSM for many tasks in high voltage engineering.

3.1.2 CSM with Discrete Charges

2D Fields

In the case of 2D fields, straight line charges of infinite length are used as simulation charges (Fig. 3.1) [1].

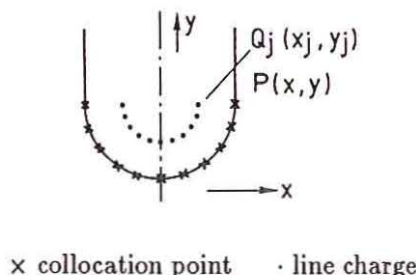


Figure 3.1: CSM for 2D fields with infinitely long line charges

The formulae for the potential coefficient and the field strength components for this type of simulation charges are well known:

$$p_j = \frac{1}{2\pi\epsilon} \ln \frac{\sqrt{(y+y_j)^2 + (x-x_j)^2}}{\sqrt{(y-y_j)^2 + (x-x_j)^2}} \quad (3.5)$$

$$\begin{aligned} E_x &= \sum_{j=1}^n \frac{\lambda_j}{2\pi\epsilon} \left[\frac{x-x_j}{(y-y_j)^2 + (x-x_j)^2} - \frac{x-x_j}{(y+y_j)^2 + (x-x_j)^2} \right] \\ E_y &= \sum_{j=1}^n \frac{\lambda_j}{2\pi\epsilon} \left[\frac{y-y_j}{(y-y_j)^2 + (x-x_j)^2} - \frac{y+y_j}{(y+y_j)^2 + (x-x_j)^2} \right] \end{aligned} \quad (3.6)$$

The terms of the image charges for the representation of an infinite earthed plane are included in these equations [1]. Since the line charges are of infinite length, the quantities to be determined are charges per unit length λ_j .

3D Fields with Axial Symmetry

The following types of simulation charges are used for 3D fields with axial symmetry [1, 2, 3, 4]:

- toroidal line charges, centred on the axis of symmetry Fig. 3.2a), usually denoted as "ring charges",
- straight line charges along the axis of symmetry Fig. 3.2b)

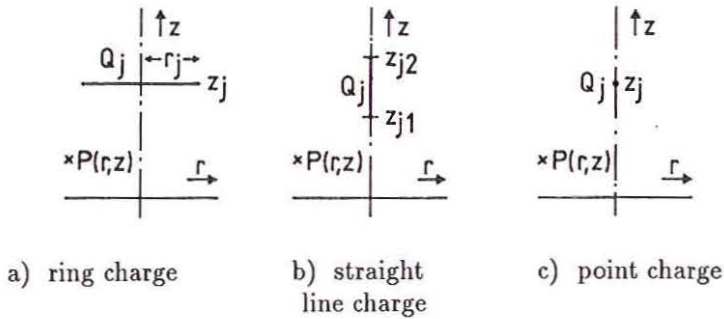


Figure 3.2: Discrete simulation charges for the calculation of 3D fields with axial symmetry

Both types having a constant charge density. Furthermore, point charges on the axis are also used (Fig. 3.2c).

The formulae for the potential coefficient p_j and the field strength components of these types of simulation charges are listed below, using the notation of Fig. 3.2. They include the image charges representing an infinite earthed plane [1]:

a) ring charges:

$$p_j = \frac{1}{4\pi\epsilon} \cdot \frac{2}{\pi} \left[\frac{K(k_1)}{\alpha_1} - \frac{K(k_2)}{\alpha_2} \right]; \quad (3.7)$$

$$\begin{aligned} E_r &= - \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon} \cdot \frac{1}{\pi r} \left\{ \frac{[r_j^2 - r^2 + (z - z_j)^2] \cdot E(k_1) - \beta_1^2 \cdot K(k_1)}{\alpha_1 \cdot \beta_1^2} - \right. \\ &\quad \left. - \frac{[r_j^2 - r^2 + (z + z_j)^2] \cdot E(k_2) - \beta_2^2 \cdot K(k_2)}{\alpha_2 \cdot \beta_2^2} \right\} \\ E_z &= - \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon} \cdot \frac{2}{\pi} \left\{ \frac{(z - z_j) \cdot E(k_1)}{\alpha_1 \cdot \beta_1^2} + \frac{(z + z_j) \cdot E(k_2)}{\alpha_2 \cdot \beta_2^2} \right\} \end{aligned} \quad (3.8)$$

with the substitutions:

$$\begin{aligned} k_1 &= \frac{2\sqrt{r_j r}}{\alpha_1}; & k_2 &= \frac{2\sqrt{r_j r}}{\alpha_2}; \\ \alpha_1 &= \sqrt{(r + r_j)^2 + (z - z_j)^2}; & \alpha_2 &= \sqrt{(r + r_j)^2 + (z + z_j)^2}; \\ \beta_1 &= \sqrt{(r - r_j)^2 + (z - z_j)^2}; & \beta_2 &= \sqrt{(r - r_j)^2 + (z + z_j)^2}; \end{aligned}$$

$K(k)$ and $E(k)$ are the complete elliptic integrals of first and second kind respectively. They can be easily evaluated by means of polynomial approximations (e.g. [5]).

b) straight line charges:

$$p_j = \frac{1}{4\pi\epsilon(z_{j2} - z_{j1})} \cdot \ln \frac{(z_{j2} - z + \gamma_1)(z_{j1} + z + \gamma_2)}{(z_{j1} - z + \delta_1)(z_{j2} + z + \delta_2)} \quad (3.9)$$

$$\begin{aligned} E_r &= - \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon(z_{j2} - z_{j1})} \cdot \frac{1}{r} \left[\frac{z - z_{j2}}{\gamma_1} - \frac{z - z_{j1}}{\delta_1} - \frac{z + z_{j1}}{\gamma_2} + \frac{z + z_{j2}}{\delta_2} \right] \\ E_z &= - \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon(z_{j2} - z_{j1})} \left[-\frac{1}{\gamma_1} + \frac{1}{\delta_1} + \frac{1}{\gamma_2} - \frac{1}{\delta_2} \right] \end{aligned} \quad (3.10)$$

where:

$$\gamma_1 = \sqrt{r^2 + (z_{j2} - z)^2}; \quad \gamma_2 = \sqrt{r^2 + (z_{j1} + z)^2};$$

$$\delta_1 = \sqrt{r^2 + (z_{j1} - z)^2}; \quad \delta_2 = \sqrt{r^2 + (z_{j2} + z)^2};$$

c) Point charges:

$$p_j = \frac{1}{4\pi\epsilon} \left[\frac{1}{\sqrt{r^2 + (z - z_j)^2}} - \frac{1}{\sqrt{r^2 + (z + z_j)^2}} \right] \quad (3.11)$$

$$E_r = \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon} \left[\frac{r}{(\sqrt{r^2 + (z - z_j)^2})^3} - \frac{r}{(\sqrt{r^2 + (z + z_j)^2})^3} \right]$$

$$E_z = \sum_{j=1}^n \frac{Q_j}{4\pi\epsilon} \left[\frac{z - z_j}{(\sqrt{r^2 + (z - z_j)^2})^3} - \frac{z + z_j}{(\sqrt{r^2 + (z + z_j)^2})^3} \right] \quad (3.12)$$

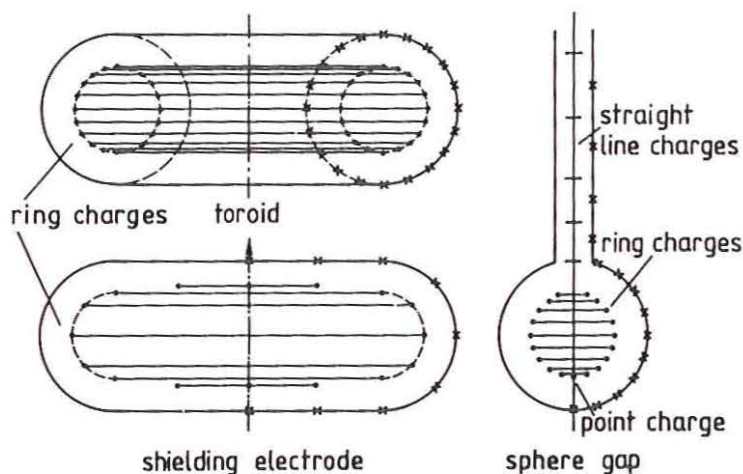


Figure 3.3: Arrangement of discrete simulation charges for the calculation of 3D fields with axial symmetry

3D Fields

Components with axial symmetry are often used in high voltage engineering (for example test transformers, voltage dividers, capacitors). However, in many cases these apparatus are arranged asymmetrically. For the calculation of the 3D electric field of these systems the CSM with discrete charges can be extended. In the same way as in the case of axial symmetry a set of point, line and ring charges can be applied for the discretization of the real surface charges. But in this case the charge density of the ring charges is either chosen as continuously variable [1, 6] or partially constant ("ring segment charges") [7] as shown in Fig. 3.4.

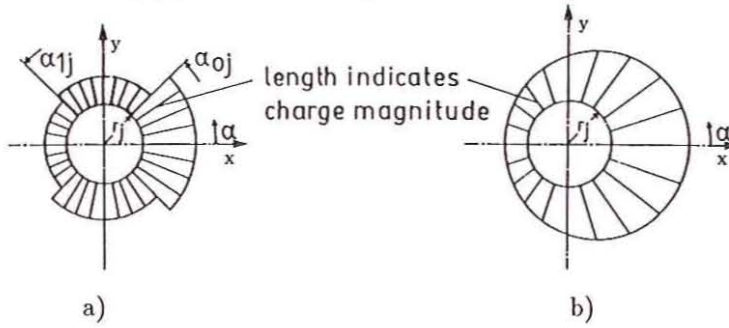


Figure 3.4: Ring charge with
a) partially constant charge density
b) continuously variable charge density

If a discretization with ring segment charges according to Fig. 3.4a is used, a collocation point on the boundary is assigned to each segment. The potential coefficient p_j and the field strength components of a segment Q_j at point $A(r, \psi, z)$ using the cylindrical coordinate system are given by:

$$p_j = \frac{1}{4\pi\epsilon(\alpha_{1j} - \alpha_{0j})} \int_{\alpha_{0j}}^{\alpha_{1j}} \frac{d\alpha}{\sqrt{D^2 - 2rr_j \cos(\psi - \alpha)}}, \quad (3.13)$$

$$\begin{aligned} E_{rj} &= \frac{Q_j}{K_j} \int_{\alpha_{0j}}^{\alpha_{1j}} \frac{r - r_j \cdot \cos(\psi - \alpha)}{F_j} d\alpha; \\ E_{\psi j} &= -\frac{Q_j}{K_j} \int_{\alpha_{0j}}^{\alpha_{1j}} \frac{r_j \cdot \sin(\psi - \alpha)}{F_j} d\alpha; \\ E_{zj} &= \frac{Q_j}{K_j} \int_{\alpha_{0j}}^{\alpha_{1j}} \frac{z - z_j}{F_j} d\alpha; \end{aligned} \quad (3.14)$$

with the abbreviations:

$$\begin{aligned} D^2 &= (z - z_j)^2 + r_j^2 + r^2; \\ K_j &= 4\pi\epsilon(\alpha_{1j} - \alpha_{0j}); \\ F_j &= \left[\sqrt{(z - z_j)^2 + r_j^2 + r^2 - 2rr_j \cdot \cos(\psi - \alpha)} \right]^3, \end{aligned} \quad (3.15)$$

and with the angles α_{0j} , α_{1j} and α according to Fig. 3.4a. The symbols r_j and z_j are the same as in the case of charges with constant charge density (Fig. 3.2a). The integrals are evaluated numerically, usually by application of Simpson's rule. More sophisticated numerical integration technique, such as Gaussian quadrature, can also be used to advantage.

If ring charges with continuously variable charge density are used for the simulation, the distribution may be divided into a constant part and several cosinusoidal (sinusoidal) harmonics with unknown peak values λ_μ , similar to Fourier analysis. This charge density distribution λ is a function of the rotation angle α , as shown in Fig. 3.4b, and as it is given by the equation:

$$\lambda_j(\alpha) = \sum_{\mu=0}^{m_H} \lambda_{j\mu} \cdot \cos(\mu\alpha) \quad (3.16)$$

In this equation m_H denotes the number of harmonics.

The total number ($m_H + 1$) of the harmonics ($\mu = 1, \dots, m_H$) and the constant component ($\mu = 0$) must be equal to the number of collocation points. The potential coefficient and the field strength components of the μ -th harmonics in point $A(r, \psi, z)$ in a cylindrical coordinate system are determined by the expression below:

$$p_{j\mu} = \frac{1}{2\pi\epsilon} \sqrt{\frac{r_j}{r}} Q_{\mu-1/2} \cdot \left(\frac{D^2}{2rr_j} \right) \cdot \cos(\mu\psi) \quad (3.17)$$

$$\begin{aligned} E_{\mu r} &= -\frac{\lambda_\mu}{2\pi\epsilon} \cdot \cos(\mu\psi) \cdot \sqrt{\frac{r_j}{r}} \cdot \left\{ \frac{1}{2r} Q_{\mu-1/2}(k_j) + \right. \\ &\quad \left. + \frac{\mu + 1/2}{D^4 - 4r_j^2 r^2} 2r_j(2r^2 - D^2) [Q_{\mu+1/2}(k_j) - k_j Q_{\mu-1/2}(k_j)] \right\} \\ E_{\mu\psi} &= \frac{\lambda_\mu}{2\pi\epsilon} \cdot \mu \cdot \sqrt{\frac{r_j}{r}} \cdot \frac{1}{r} \cdot Q_{\mu-1/2}(k_j) \cdot \sin(\mu\psi) \\ E_{\mu z} &= -\frac{\lambda_\mu}{2\pi\epsilon} \cdot \cos(\mu\psi) \cdot \sqrt{\frac{r_j}{r}} \cdot \frac{\mu + 1/2}{D^4 - 4r_j^2 r^2} \cdot 4r_j r \cdot (z - z_j) \cdot \\ &\quad \cdot [Q_{\mu+1/2}(k_j) - k_j \cdot Q_{\mu-1/2}(k_j)] \end{aligned} \quad (3.18)$$

with $k_j = D^2/(2rr_j)$, D^2 according to equation (3.15.) and $Q_{\mu+1/2}$ as the Legendre functions of the second kind of the order $\mu - 1/2$ and $\mu + 1/2$ respectively, so-called torus functions.

The resulting system of linear equations is solved for the charge density harmonics $\lambda_{j\mu}$. The resulting field strength in any direction is calculated by addition of the parts which originate from constant and periodically variable charge components, e.g. taking one ring charge:

$$E_r = \sum_{\mu=0}^{m_H} E_{\mu r} \quad (3.19)$$

3.1.3 Arrangements with More Dielectrics

The procedures described above are applicable for systems with only one dielectric, for example electrode systems in air or SF_6 . In high voltage engineering very often fields with more than one dielectric have to be calculated. In principle the CSM with discrete charges also can be applied to these fields [1, 8].

A multidielctric arrangement should be divided into regions. Each region is homogeneous with regard to its material properties, i.e. it consists of only one dielectric. For the collocation points placed on the regions boundaries (see model arrangement, Fig. 3.5a), the discrete charges are simulated as follows:

1. for each point on the electrode surface one charge is placed in front of the adjacent region,
2. for each point on the interface between regions two charges are placed in front of the interface, each on the opposite side.

The number of unknown charge values match the number of equations when taking into account that for the interface between two regions R_i and R_j , the following two conditions must be fulfilled:

1. the continuity of the potential:

$$\Phi_i = \Phi_j \quad (3.20)$$

2. the relation of the normal field strength components:

$$\epsilon_i E_{ni} = \epsilon_j E_{nj} \quad (3.21)$$

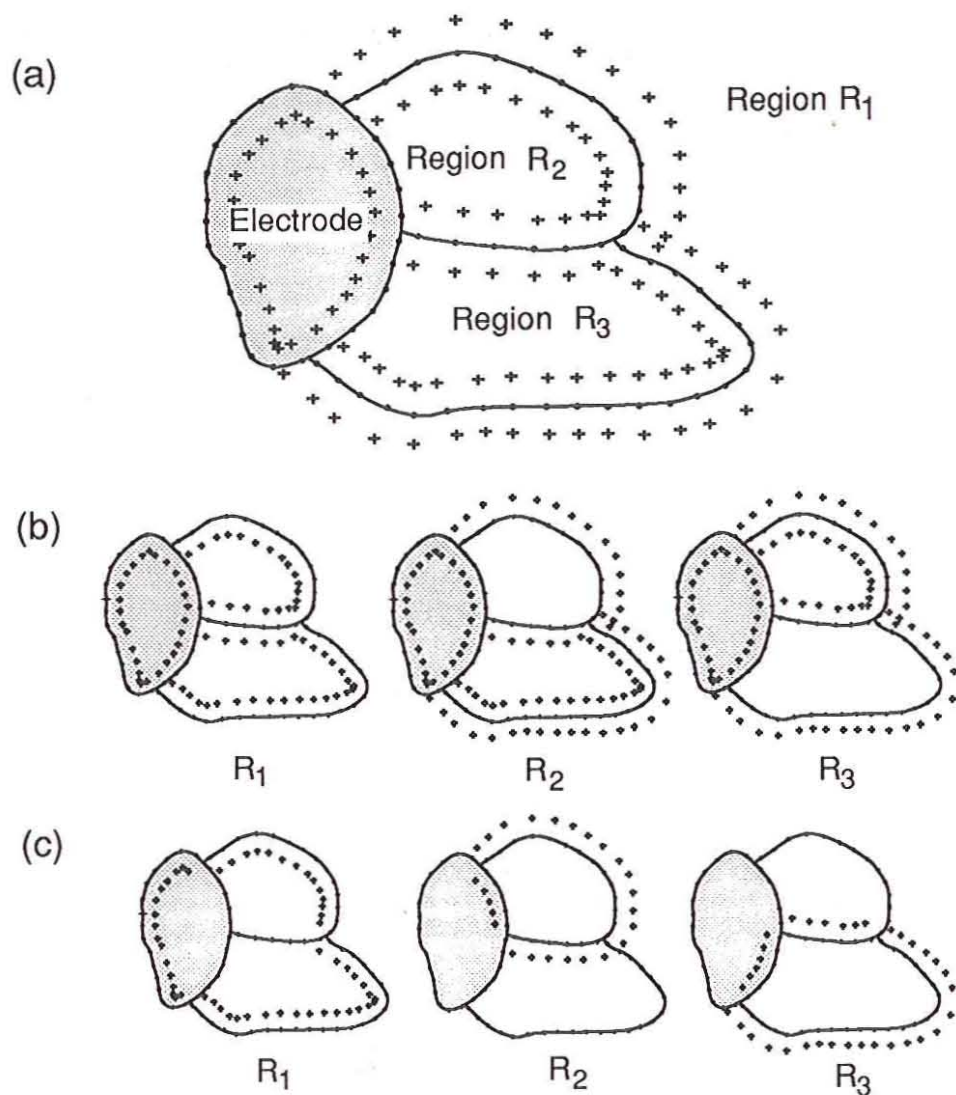


Figure 3.5: Charge simulation in a model arrangement with more dielectrics (regions):

- a) charges simulated in the whole arrangement
- b) surface-oriented CSM: set of charges used for field calculations in regions R_1 , R_2 and R_3
- c) region-oriented CSM: set of charges used for field calculations in regions R_1 , R_2 and R_3

with potentials Φ_i and Φ_j , the normal field strength components E_{ni} and E_{nj} and with the scalar dielectric constants ϵ_i and ϵ_j of the region R_i and R_j respectively.

There are two different approaches when calculating field within regions. One should distinguish between surface-oriented and region-oriented CSM.

Surface-oriented CSM

For the calculation of potential and field strength all charges are regarded except those which are arranged inside the region of consideration, Fig. 3.5b. For instance, if point $A(r, \psi, z)$ belongs to a region R_i with the dielectric constant ϵ_i , the charges belonging to this region are disregarded:

$$\Phi(r, \psi, z) = \sum_{Q_k \notin R_i} p_k(r, \psi, z) \cdot Q_k \quad (3.22)$$

$$\vec{E}(r, \psi, z) = - \sum_{Q_k \notin R_i} [\nabla p_k(r, \psi, z)] \cdot Q_k \quad (3.23)$$

where $Q_k \notin R_i$ denotes charges placed outside the region R_i .

These formulae can also be applied to each collocation point $C(r_c, \psi_c, z_c)$ on the region boundaries when building the equation system:

- for a point on the electrode boundary:

$$\sum_{Q_k \notin R_i} p_k(r_c, \psi_c, z_c) \cdot Q_k = \Phi_e \quad (3.24)$$

with the constant potential Φ_e on the electrode. (If the potential Φ_e is unknown, i.e. if the electrode is on a floating potential, an additional equation has to be formulated. For the surface-oriented CSM, the condition is usually applied that the sum of charges assigned to the collocation points on the electrode equals zero.)

- for a point on the dielectric boundary

$$\sum_{Q_k \notin R_i} p_k(r_c, \psi_c, z_c) \cdot Q_k = \sum_{Q_k \notin R_j} p_k(r_c, \psi_c, z_c) \cdot Q_k \quad (3.25)$$

$$\epsilon_i \sum_{Q_k \notin R_i} [\vec{n} \cdot \nabla p_k(r_c, \psi_c, z_c) \cdot Q_k] = \epsilon_j \sum_{Q_k \notin R_j} [\vec{n} \cdot \nabla p_k(r_c, \psi_c, z_c) \cdot Q_k] \quad (3.26)$$

The surface-oriented method was successfully applied to arrangements with 2 dielectrics [1, 8]. It is also possible to obtain good results for systems with more than 2 dielectrics. However, the surface-oriented CSM is rather sensitive regarding the positions of simulation charges: they should be placed close to the corresponding surfaces to avoid overlapping with charges simulated for other surfaces.

Region-oriented CSM

The charges should be formed into sets so that for each region an unique set of charges is arranged. Each of these sets surrounds the appropriate region, Fig. 3.5c. For the calculation of potential and field in a region R_i the following formulas should be applied:

$$\Phi(r, \psi, z) = \sum_{Q_k \in \{R_i\}} p_k(r, \psi, z) \cdot Q_k \quad (3.27)$$

$$\vec{E}(r, \psi, z) = - \sum_{Q_k \in \{R_i\}} [\nabla p_k(r, \psi, z)] \cdot Q_k \quad (3.28)$$

where $Q_k \in \{R_i\}$ denotes charges surrounding region R_i .

Using (3.27) and (3.28) the equations similar to (3.24) - (3.26) can be written for collocation points on the region boundaries: the only difference is the expression under the Σ . It results, however, in the sparse matrix of the equation system. This is one of fundamental differences between the region-oriented CSM and other integral methods. (However, the matrix for the surface-oriented CSM is also not quite "full". But the number of elements which are equal to zero is usually less than 50% of the total number of matrix elements.)

In the region-oriented CSM charges influencing the field in one region are not used for the field calculations in other regions. This fact is very useful for positioning the charges. Let us consider a collocation point on a boundary between region R_i and R_j . For this point two charges have to be assigned to the appropriate regions. It is required that the charge assigned to the region R_i is not placed inside of the region R_i . However, it is not necessary that this charge is placed within the region R_j . The same rule can be applied for charges simulated for the electrode surface: they need not to be placed inside the electrode. This opens additional possibilities for calculating fields near very thin conducting foils.

In the region-oriented CSM - in contrast to other methods - one does not distinguish between free and bound charge. This follows from the fact that charges are assigned to regions rather than to surfaces, and this has important consequences, for example when solving problems with floating potentials. The flux through an electrode on floating potential cannot be calculated as the sum of charges assigned to the electrode. The flux can only be obtained by the integration of the electric displacement over the electrode surface according to the Gauss's law.

3.1.4 Application Aspects

Position of the Discrete Charges

In applications of CSM with discrete charges, the arrangement of the simulation charges and collocation points is important. An empirical criterion which has proven as very effective is the assignment factor:

$$f_a = \frac{a_2}{a_1} \quad (3.29)$$

with the distance a_1 between two successive collocation points and the distance a_2 between a collocation point and the corresponding charge (Figure 3.6a). In order to avoid distances between charges and collocation points which are too small, in the case of curved contours the definition of a curvature criterion according to Fig. 3.6b is useful [1]:

$$\rho_{1/0} = r \left[\sqrt{1 + \left(f_a \cdot \frac{a_1}{r}\right)^2} \mp f_a \cdot \frac{a_1}{r} \right] \quad (3.30)$$

ρ_1 is valid for convex curvature (negative sign), ρ_0 for concave curvature (positive sign). For dielectric interfaces, both cases must be used accordingly.

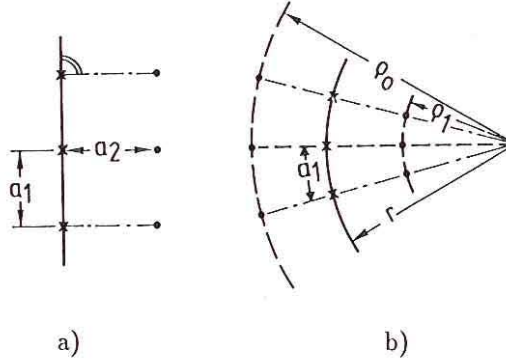


Figure 3.6: Arrangement of simulation charges

Experience shows that the assignment factor should be between 1,0 and 2,0. If f_a lies within these limits the accuracy of the calculation is not very much influenced by the position of the simulation charges. Another possibility to determine the position of the simulation charges is by introducing optimization procedures, for example the Fletcher algorithm [9], or the application of knowledge based systems [25].

Check of Accuracy

For the check of accuracy usually the following criteria are applied:

- (1) Deviation of the resulting equipotential line $\Phi = \Phi_B$ from the given boundary,
- (2) Deviation of the calculated field strength components on electrode surfaces from the normal direction (Fig. 3.7),
- (3) Relation between field strength, normal derivative of field strength and curvature radius, especially for electrodes with small radius of curvature (formula of Spielrein [20]):

$$\frac{\partial E}{\partial n} \cdot \frac{1}{E} = - \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \quad (3.31)$$

with the radii of curvature ρ_1 and ρ_2 of two perpendicular cross sections.

For systems with more dielectrics the following criteria on the dielectric boundaries can be used additionally [8]:

- (4) Continuity of potential,
- (5) Continuity of the tangential components of the field strength,
- (6) Relation of the normal components of the field strength.

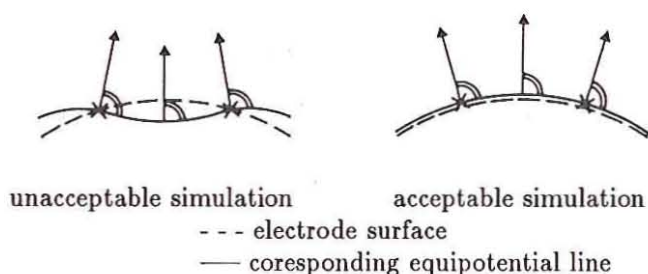


Figure 3.7: Check of the accuracy by the direction of field strength vector $\vec{E}$

3.1.5 Charge Simulation Method with Area Charges

While CSM with discrete charges operates with charge distributions arranged outside the domain of consideration, CSM with area charges uses distributions which are located directly on the boundaries. This kind of CSM may contribute both to the family of CSM and the family of BEM. For application in high voltage engineering, the CSM with area charges was developed by Singer [10, 11]. It is described here for the calculation of asymmetrical 3D fields with axial symmetric components as shown in section 3.1.2 (3D fields).

The real surface charge of an electrode or a dielectric interface is subdivided into T area charges with the area charge density σ_j and the area A_j , Figure 3.8. Then the potential in a field point $P(r, \vartheta, \psi)$ in spherical coordinate system is the sum of the potentials of all area charges:

$$\Phi(r, \vartheta, \psi) = \frac{1}{4\pi\epsilon} \sum_{j=1}^T \sigma_j \int_{A_j} \frac{dA}{s} \quad (3.32)$$

with the distance s between the point P and the area element dA on the area charge.

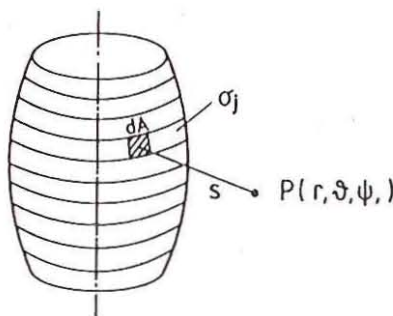


Figure 3.8: Discretization with area charges

For the calculation of asymmetrical 3D fields with axisymmetrical components the area charge density σ_j is a function of the rotation angle α . This function can be expressed in the same way as it was shown in section 3.1.2 for ring charges by cosinusoidal harmonics with the unknown peak values of the area charge density $\sigma_{j\mu}$:

$$\sigma_j = \sum_{\mu=0}^{m_H} \sigma_{j\mu} \cos(\mu\alpha) \quad (3.33)$$

with m_H the number of harmonics.

In order to avoid discontinuities between two successive charge elements, a dependence on the coordinate r is introduced into the approximation formula. If this dependence is formulated as a polynomial distribution of the charge density, σ_j can be expressed by the equation:

$$\sigma_j = \sum_{\mu=0}^{m_H} \sum_{\nu=0}^{n_H} \sigma_{j\mu\nu} \left(\frac{r}{r_0} \right)^\nu \cdot \cos(\mu\alpha) \quad (3.34)$$

r_0 is an arbitrarily related radius, n_H is the maximum degree of the polynomials. For each simulation charge we get $(m_H + 1)(n_H + 1)$ unknown charge density components $\sigma_{j\mu\nu}$. In order to determine these unknowns one needs the same number of collocation points assigned to each area charge. The setup and the solution of the equation system is the same as for the CSM with discrete charges.

The calculation of the potential functions of the area charges is more difficult than the calculation of potential functions of discrete charges. Usually numerical integration methods must be applied. In comparison with CSM with discrete charges the CSM with area charges needs more computation time. Therefore it is proposed to use it in combination with discrete charges. Area charges should be applied to thin plates, strip conductors or foils, as they are used in bushings or cable terminations [10].

The application of area charges is advantageous for the simulation of dielectric boundaries. In this case only the relation of the normal field strength components must be regarded as boundary conditions:

$$\frac{E_{nII}}{E_{nI}} = \frac{\epsilon_I}{\epsilon_{II}} \quad (3.35)$$

In contrast to the simulation with discrete charge the condition of potential continuity is automatically fulfilled. For every collocation point on the dielectric boundary only one equation is necessary and therefore the resulting matrix is smaller.

3.2 Boundary Element Method

3.2.1 Mathematical Model

The electrostatic field of the scalar potential function can be described using the integral equations (Fredholm's integral equation of first and second kind) with a Green's function kernel.

The basis for establishing of the system of integral equations is the introduction of equivalent surface charges distributed in vacuum, as follows:

- over the boundary interface between two dielectrics, taking into account the bound surface charge,
- over the conductor - dielectric boundary interface, where the conducting body has known potential,
- over the conductor - dielectric boundary interface, where the total quantity of the charge on the conductor is known, and is possibly zero.

The distribution function of the introduced equivalent surface charges over the marked boundary interfaces, is determined such that following conditions can be satisfied:

- boundary conditions on the boundaries between two dielectrics,
- prescribed values for the potential function on the conductor surfaces with known potential are maintained,
- prescribed values for the total quantity of surface charges on the conductors with unknown potential ("floating potential") are imposed.

The electrical field in the observed space of interest can now be considered as the result of the presence of the equivalent surface charges. The charges are considered to be the "sources" of the electric field.

Let us introduce following notation:

S_p - boundary surface conductor - dielectric, $p = 1, \dots, P$;

S_d - boundary surface dielectric - dielectric, $d = 1, \dots, D$;

M - index of the point of field source lying on the conductor surface;

N - index of the point of field source lying on a dielectric - dielectric boundary surface;

I - index of an arbitrary point lying on the conductor surface;

J - index of an arbitrary point lying on the dielectric - dielectric interface;

σ - the function of the distribution of equivalent surface charges.

If the functions of equivalent surface charges are known, then, for an arbitrary point I on the surface of the conductor, the potential is calculated according to the following relation:

$$\Phi_I = \frac{1}{4\pi\epsilon_0} \sum_{p=1}^P \int_{S_p} \frac{\sigma(M)}{r_{MI}} dS_p + \frac{1}{4\pi\epsilon_0} \sum_{d=1}^D \int_{S_d} \frac{\sigma(N)}{r_{NI}} dS_d \quad (3.36)$$

$$I \in S_p, p = 1, \dots, P.$$

Equation (3.36) represents a Fredholm integral equation of the first kind.

In a case where the potential of some conductor is not known, but the total quantity of charge Q on that conductor is known, then the equation representing the total quantity of charges is used together with the equation (3.36):

$$Q_p = \int_{S_p} \epsilon_0 \epsilon_p E_{np} dS_p \quad (3.37)$$

The normal component of the electric field strength of an arbitrary point J on the boundary interface dielectric - dielectric can be calculated as follows:

$$E_{nJ} = \sum_{p=1}^P E_{nJ}^{(p)} + \sum_{d=1}^D E_{nJ}^{(d)} = E_{nJ}^{(0)} + E_{nJ}^{(J)} \quad (3.38)$$

where :

$E_{nJ}^{(0)}$ - normal component of the electrical field strength of all the sources, except the sources along the boundary surface where point J is found,

$E_{nJ}^{(J)}$ - normal component of the electrical field strength of all the sources along the boundary surface where point J is found.

$$\begin{aligned} E_{nJ}^{(0)} = & \frac{1}{4\pi\epsilon_0} \sum_{p=1}^P \int_{S_p} \frac{\sigma(M) \cos(\vec{r}_{MJ}, \vec{n}_J) dS_p}{r_{MJ}^2} + \\ & + \frac{1}{4\pi\epsilon_0} \sum_{\substack{d=1 \\ d \neq d_J}}^D \int_{S_d} \frac{\sigma(N) \cos(\vec{r}_{NJ}, \vec{n}_J) dS_d}{r_{NJ}^2} \end{aligned} \quad (3.39)$$

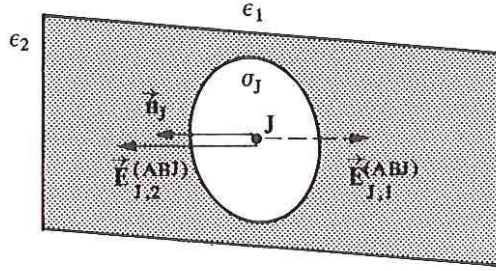
The normal component of the electrical field strength at point J , caused by the sources along the boundary surface d , is calculated as follows:

$$E_{nJ}^{(J)} = \frac{1}{4\pi\epsilon_0} \int_{S_{dJ}} \frac{\sigma(N) \cos(\vec{r}_{NJ}, \vec{n}_J) dS_{dJ}}{r_{NJ}^2} \quad (3.40)$$

Here we notice that for $N \rightarrow J$ the integral (3.40) has singularity.

In order to avoid the singularity, the small area around point J , area S_{AB} , Fig. 3.9, will be excluded from the integral (3.40), and the field caused by the sources on that area S_{AB} will be calculated separately.

$$E_{nJ}^{(J)} = E_{nJ}^{(J-)} + E_{nJ}^{(ABJ)} \quad (3.41)$$

Figure 3.9: The small area around singular point J

where:

$$E_{nJ}^{(J-)} = \frac{1}{4\pi\epsilon_0} \int_{S_{dJ-}} \frac{\sigma(N) \cos(\vec{r}_{NJ}, \vec{n}_J) dS_{dJ-}}{r_{NJ}^2} \quad S_{dJ-} = S_{dJ} \setminus S_{AB} \quad (3.42)$$

The area S_{AB} can be regarded as a small flat circular disc, so that its field strength in center is normal to the disc surface. In this formulation, dielectrics are modelled by the bound surface charge at their interfaces, and flux continuity is imposed at collocation points along these boundaries. As such, relative permittivities are postponed until the matrix is formulated. Close to the surface, the field strengths on the both sides of the disk are:

$$E_{J,2}^{(ABJ)} = -E_{J,1}^{(ABJ)} = \frac{\sigma_J}{2\epsilon_0} \quad (3.43)$$

and the total normal components of the field strength at point J on both sides of the boundary surface, calculated in the direction of the normal vector with orientation to the side "2", are:

$$\begin{aligned} E_{nJ,1} &= E_{nJ}^{(0)} + E_{nJ}^{(J-)} - \frac{\sigma_J}{2\epsilon_0} \\ E_{nJ,2} &= E_{nJ}^{(0)} + E_{nJ}^{(J-)} + \frac{\sigma_J}{2\epsilon_0} \end{aligned} \quad (3.44)$$

If σ_S is the bound surface charge on the boundary dielectric - dielectric, then on the basis of the boundary conditions on the interface of two dielectrics the Gauss' law jump condition can be written:

$$\begin{aligned} \epsilon_0\epsilon_2 E_{nJ,2} - \epsilon_0\epsilon_1 E_{nJ,1} &= \sigma_{sJ} \\ \epsilon_0\epsilon_2 \left[E_{nJ}^{(0)} + E_{nJ}^{(J-)} + \frac{\sigma_J}{2\epsilon_0} \right] - \epsilon_0\epsilon_1 \left[E_{nJ}^{(0)} + E_{nJ}^{(J-)} - \frac{\sigma_J}{2\epsilon_0} \right] &= \sigma_{sJ} \end{aligned} \quad (3.45)$$

so we obtain:

$$\sigma_J = \frac{2\sigma_{sJ}}{\varepsilon_1 + \varepsilon_2} + 2\varepsilon_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \left[E_{nJ}^{(0)} + E_{nJ}^{(J-)} \right] \quad (3.46)$$

$$J \in S_d, d = 1, \dots, D.$$

The equation (3.46) represents the second order Fredholm integral equation. For its solution, relations (3.39) and (3.42) together with $S_{AB} \rightarrow 0$, need to be used.

3.2.2 Discrete Mathematical Model

For the numerical solution of Fredholm integral equations, the Boundary Element Method (BEM) is successfully used. According to the boundary element method, where an approximate distribution of equivalent surface charges is determined, boundary surfaces are approximated using a finite number of subareas - boundary surface elements - often in the form of a curvilinear triangle or curvilinear quadrangle. The boundary element itself is described using the shape functions of a different order, depending how precisely the boundary element needs to be defined. In a great number of cases, boundary surfaces are regular surfaces of the second order, which can be fully and simply described analytically, as a whole and in parts, in an appropriate local and arbitrary absolute coordinate system. This specific case of Boundary Element Method is called Partial Element Method (PEM).

The problem of solving of integral equations is reduced to a problem of determining a set of functions of the equivalent surface charges defined on each boundary element:

$$\Phi_I = \frac{1}{4\pi\varepsilon_0} \sum_{e=1}^E \int_{S_e} \frac{\sigma_e dS_e}{r_{GI}} \quad (3.47)$$

$$I \in S_p, p = 1, \dots, P; G \in S_e, e = 1, \dots, E$$

$$\sigma_J = \frac{2\sigma_{sJ}}{\varepsilon_1 + \varepsilon_2} + 2\varepsilon_0 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{1}{4\pi\varepsilon_0} \sum_{e=1}^E \int_{S_e} \frac{\sigma_e \cos(\vec{r}_{GJ}, \vec{n}_J) dS_e}{r_{GJ}^2} \quad (3.48)$$

$$J \in S_d, d = 1, \dots, D; G \in S_e (G \neq J), e = 1, \dots, E;$$

where E is the total number of boundary elements.

The numerical solution of the equations (3.47) and (3.48), calls for the introduction of suitable functions for approximating the function $\sigma(S)$. For that purpose, it is best to use so called basis functions (similar to the method of finite elements for approximation of the scalar function of the potential) of a different order.

If the values of the function in m of the appropriately selected points of the boundary element are known, then the function value at an arbitrary point of the element can be calculated using appropriate basis functions:

$$f = \sum_{j=1}^m N_j f_j \quad (3.49)$$

The shape of the basis function N_j depends on the polynomial order. In most cases, it is enough to use polynomials of first or second order. In these cases, it is very suitable to define basis functions in a separate local coordinate system, so called L -coordinate system. For example, for a curvilinear triangle-shaped boundary element, a linear approximation is defined as follows:

$$f = L_1 f_1 + L_2 f_2 + L_3 f_3 \quad (3.50)$$

where $L_1 + L_2 + L_3 = 1$, and (f_1, f_2, f_3) are function values in the vertices of the simplex triangle. The relation (3.50) can be expressed in the matrix form:

$$f = [L_1 L_2 L_3] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = [N^{(1)}][f^{(1)}] .$$

This method of expressing of the function values of an element can be considered as general, whatever the approximation order is. Moreover, the dimensions of introduced matrix and the form of the basis functions will depend on the approximation order. Thus:

$$f = [N^{(e)}][f^{(e)}] , \quad (3.51)$$

where following symbols are introduced:

$[N^{(e)}]$ - transformation matrix, where the row matrix contains basis functions of elements,

$[f^{(e)}]$ - column matrix of the function values in the specially selected element points,

(e) - order of the polynomial approximation.

The use of linear approximation for the unknown function of the equivalent surface charge, has proven precise enough:

$$\sigma = L_1\sigma_1 + L_2\sigma_2 + L_3\sigma_3 \quad ; \quad L_1 + L_2 + L_3 = 1 \quad (3.52)$$

The boundary elements themselves are approximated by second order functions, or they are represented by partial elements.

By replacing (3.52) in (3.47) and (3.48), and assuming that the points (I, J) coincide with the vertices of the boundary elements, the set of integral equations is translated into a system of linear algebraic equations:

$$[A][\sigma] = [B] \quad (3.53)$$

Matrix coefficients $[A]$ and $[B]$ are calculated using the relations (3.47), (3.48), (3.52), taking into account the way the boundary surfaces are defined, that is, a curvilinear boundary element or partial boundary element.

3.2.3 Numerical Procedures

3D FIELDS

Boundary Surfaces Approximated by Curvilinear Triangles

Let us choose the triangular boundary element, Fig. 3.10a and its transformation in the L -coordinate system (master triangle), Fig. 3.10b. The main nodes are N_1, N_2, N_3 , while the auxiliary are marked as N_4, N_5, N_6 . The coordinates of an arbitrary point G inside the element, in a case of a second order approximation are calculated using the following relations:

$$\begin{aligned} x &= \sum_{j=1}^6 N_j^{(2)} x_j \\ y &= \sum_{j=1}^6 N_j^{(2)} y_j \\ z &= \sum_{j=1}^6 N_j^{(2)} z_j \end{aligned} \quad (3.54)$$

where:

$$\begin{aligned} N_1^{(2)} &= L_1(2L_1 - 1) & N_2^{(2)} &= L_2(2L_2 - 1) & N_3^{(2)} &= L_3(2L_3 - 1) \\ N_4^{(2)} &= 4L_1L_2 & N_5^{(2)} &= 4L_2L_3 & N_6^{(2)} &= 4L_3L_1 \end{aligned}$$

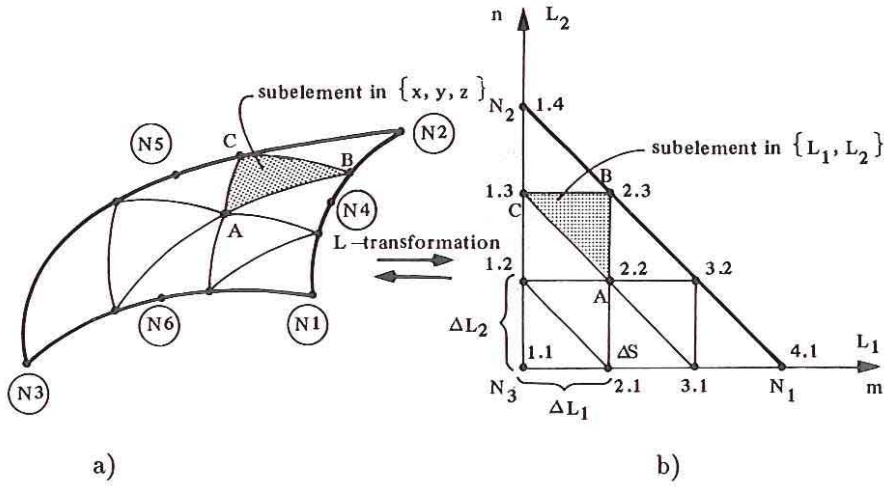


Figure 3.10: "L" - transformation of the boundary element

The surface charges function on an element is calculated using the linear approximation:

$$\sigma_e(L_1, L_2) = L_1\sigma_1 + L_2\sigma_2 + L_3\sigma_3$$

The contribution of the observed element to the potential in point I is:

$$\Phi_{eI} = \frac{1}{4\pi\epsilon_0} \int_0^1 \int_0^{1-L_1} \frac{L_1\sigma_1 + L_2\sigma_2 + (1-L_1-L_2)\sigma_3}{r_{GI}(L_1, L_2)} |n(L_1, L_2)| dL_1 dL_2 \quad (3.55)$$

where $|n(L_1, L_2)|$ is the Jacobean of the transformation. The relation (3.55) determines the row elements of the discretization matrix $[A]$ which corresponds to I Fredholm integral equation:

$$\Phi_{eI} = a_{I1}\sigma_1 + a_{I2}\sigma_2 + a_{I3}\sigma_3$$

where:

$$\begin{aligned} a_{I1} &= \frac{1}{4\pi\epsilon_0} \int_0^1 \int_0^{1-L_1} \frac{L_1 |n(L_1, L_2)| dL_1 dL_2}{r_{GI}[G(L_1, L_2), I]} \\ a_{I2} &= \frac{1}{4\pi\epsilon_0} \int_0^1 \int_0^{1-L_1} \frac{L_2 |n(L_1, L_2)| dL_1 dL_2}{r_{GI}[G(L_1, L_2), I]} \\ a_{I3} &= \frac{1}{4\pi\epsilon_0} \int_0^1 \int_0^{1-L_1} \frac{(1-L_1-L_2) |n(L_1, L_2)| dL_1 dL_2}{r_{GI}[G(L_1, L_2), I]} \end{aligned} \quad (3.56)$$

The Gauss Radau method was employed to solve numerically the integral in (3.56). If each side of the master triangle is divided in N parts, we get N^2 sub-elements whose vertical sides' lengths are $\Delta L_1 = 1/N$, $\Delta L_2 = 1/N$. On each element $0, 5(N+1)(N+2)$ of the cut points are formed (local nodes), including the main nodes as well. In the L -coordinate system, the coordinates of the local nodes are determined as follows:

$$[L_1(m), L_2(n)] = \left[\frac{m-1}{N}, \frac{n-1}{N} \right]; \quad 1 \leq m, n \leq N+1.$$

Using the integration through the sub-elements, the relations (3.56) acquire the following form:

$$\begin{aligned} a_{I1} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{L_1(m)|n[L_1(m), L_2(n)]|}{N^2 r_{GI}[G(L_1(m), L_2(n)), I]} \\ a_{I2} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{L_2(n)|n[L_1(m), L_2(n)]|}{N^2 r_{GI}[G(L_1(m), L_2(n)), I]} \\ a_{I3} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{[1 - L_1(m) - L_2(n)]|n[L_1(m), L_2(n)]|}{N^2 r_{GI}[G(L_1(m), L_2(n)), I]} \end{aligned} \quad (3.57)$$

$g_{mn} = 1/6$ - for the points in the triangle vertices;

$g_{mn} = 1/2$ - for the points on the sides of a triangle;

$g_{mn} = 1$ - for the points inside the triangle.

The relations (3.57) are applicable in the case where the observed point I does not coincide with any of the main nodes of the boundary element. In the opposite situation $r_{GI} = 0$, that is, the integrals in (3.56) are singular. The singular case can be solved using the relation (3.57), if we exclude the sub-element whose one point is the examined singular node (Fig. 3.11). Using the appropriate numeration, the singular node could be marked as the node $N3$. In that case, in relations (3.57) the node $(1, 1)$ of the sub-element should be excluded and the weight factors g_{12} and g_{21} corrected. In that way, for a singular case we can write:

$$a_{I1} = a'_{I1} + \Delta a_{I1} \quad a_{I2} = a'_{I2} + \Delta a_{I2} \quad a_{I3} = a'_{I3} + \Delta a_{I3}$$

The parts a'_{I1} , a'_{I2} , a'_{I3} are calculated according to the relations (3.57) while excluding the points $(m, n) = (1, 1)$ and putting $g_{12} = g_{21} = 1/3$.

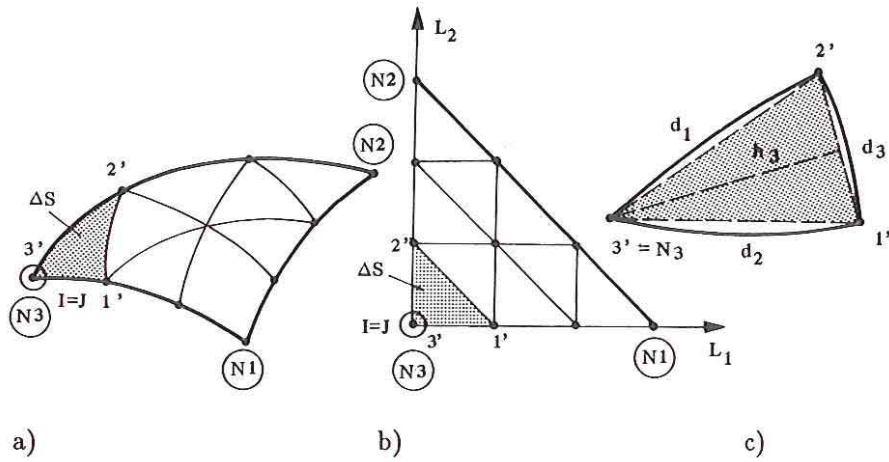


Figure 3.11:

- a) discretization of the boundary element
- b) discretization of the "master" element
- c) singular subelement

The parts Δa_{I1} , Δa_{I2} , Δa_{I3} are calculated analytically according to the relations:

$$\begin{aligned}
 \Delta a_{I1} &= \frac{1}{16\pi\epsilon_0 d_3^4 N^4} [h_3(d_1^2 + d_3^2 - d_2^2) \cdot \ln \frac{d_1 + d_2 + d_3}{d_1 + d_2 - d_3} - 2d_3^2(d_1 - d_2)] \\
 \Delta a_{I2} &= \frac{1}{16\pi\epsilon_0 d_3^4 N^4} [h_3(d_2^2 + d_3^2 - d_1^2) \cdot \ln \frac{d_1 + d_2 + d_3}{d_1 + d_2 - d_3} - 2d_3^2(d_2 - d_1)] \\
 \Delta a_{I3} &= (N-1)\Delta a_{I1} + (N-1)\Delta a_{I2} + \frac{h_3(N-0,5)}{16\pi\epsilon_0 d_3 N^3} \ln \frac{d_1 + d_2 + d_3}{d_1 + d_2 - d_3}
 \end{aligned} \quad (3.58)$$

In a similar way a discrete form of the second order Fredholm integral equation can be written:

$$\sigma_J = \frac{2\sigma_{sJ}}{\epsilon_1 + \epsilon_2} + \frac{\lambda}{4\pi\epsilon_0} \sum_{e=1}^E \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} \frac{\vec{r}_{GJ}[J, G(L_1(m), L_2(n))] \cdot \vec{n}_J^0}{N^2 r_{GJ}^3[J, G(L_1(m), L_2(n))]} \cdot g_{mn} \cdot |n[L_1(m), L_2(n)]| \cdot \sigma_e[L_1(m), L_2(n)] \quad (3.59)$$

$$\lambda = 2\epsilon_0 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$

For the case of a singular element in the relation (3.59), the part which corresponds to the sub-element with the singular node, that is, the part $(m, n) = (1, 1)$, should be omitted. If the number of sub-elements is relatively small, the avoiding of the whole sub-element could lead to the decreasing of the accuracy of the calculation. Therefore, a separate procedure is introduced where the singular node of the sub-element is isolated with a small enough circular surface. The contribution of the rest of the sub-element to the equation (3.59) is then determined analytically.

Boundary Surfaces Defined by the Partial Elements

Partial elements are introduced for the boundary surfaces of the following forms:

- spherical surface,
- conical surface,
- cylindrical surface,
- paraboloid surface,
- hyperboloid surface,
- torus surface,
- flat circular surface.

The point coordinates on each of the above given surfaces, could be defined in an suitable local coordinate system. On these boundary surfaces, square or triangular boundary elements could be placed. In both cases, the surface charge density along the element is approximated using linear base functions:

- for a square element, Fig. 3.12:

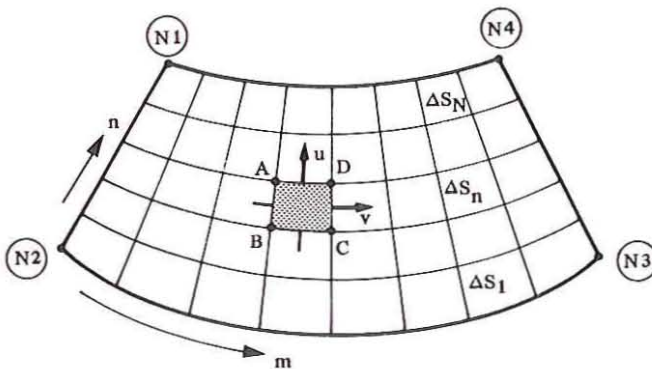


Figure 3.12: Partial "square" boundary element

$$\begin{aligned}\sigma_e(m, n) = & 0,25(1+u_n)(1-v_m)\sigma_1 + 0,25(1-u_n)(1-v_m)\sigma_2 + \\ & + 0,25(1-u_n)(1+v_m)\sigma_3 + 0,25(1+u_n)(1+v_m)\sigma_4 \quad (3.60)\end{aligned}$$

- for a triangular element, Fig. 3.13:

$$\sigma_e(m, n) = \frac{m-1}{N}\sigma_1 + \frac{n-1}{N}\sigma_2 + \left(1 - \frac{m-1}{N} - \frac{n-1}{N}\right)\sigma_3 \quad (3.61)$$

Using the procedure of integration through sub-elements, as it was explained in the part relating to the approximation of the boundary surface with the help of curvilinear triangles, the following relations are derived:

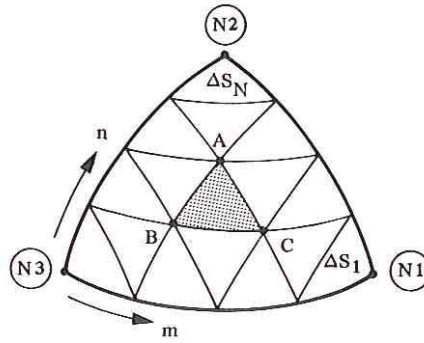


Figure 3.13: Partial "triangle" boundary element

- contribution of the square element to the potential at point I :

$$\Phi_{eI} = a_{I1}\sigma_1 + a_{I2}\sigma_2 + a_{I3}\sigma_3 + a_{I4}\sigma_4$$

$$\begin{aligned}a_{I1} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{M+1} g_{mn} \frac{0,25(1+u_n)(1-v_m)}{r_{GI}[G(u_n, v_m), I]} \\ a_{I2} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{M+1} g_{mn} \frac{0,25(1-u_n)(1-v_m)}{r_{GI}[G(u_n, v_m), I]} \\ a_{I3} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{M+1} g_{mn} \frac{0,25(1-u_n)(1+v_m)}{r_{GI}[G(u_n, v_m), I]} \\ a_{I4} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{M+1} g_{mn} \frac{0,25(1+u_n)(1+v_m)}{r_{GI}[G(u_n, v_m), I]} \quad (3.62)\end{aligned}$$

where:

$$\begin{aligned}
g_{mn} &= 0, 25\Delta S_1 & (n=1) \wedge (m=1 \vee m=M+1) \\
g_{mn} &= 0, 50\Delta S_1 & (n=1) \wedge (m \neq 1 \wedge m \neq M+1) \\
g_{mn} &= 0, 25\Delta S_N & (n=N+1) \wedge (m=1 \vee m=M+1) \\
g_{mn} &= 0, 50\Delta S_N & (n=N+1) \wedge (m \neq 1 \wedge m \neq M+1) \\
g_{mn} &= 0, 25(\Delta S_n + \Delta S_{n-1}) & (m=1 \vee m=M+1) \wedge (n \neq 1 \wedge n \neq N+1) \\
g_{mn} &= 0, 50(\Delta S_n + \Delta S_{n-1}) & (m \neq 1 \vee m \neq M+1) \wedge (n \neq 1 \wedge n \neq N+1)
\end{aligned}$$

- contribution of the triangular element to the potential in the point I :

$$\Phi_{eI} = a_{I1}\sigma_1 + a_{I2}\sigma_2 + a_{I3}\sigma_3$$

$$\begin{aligned}
a_{I1} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{m-1}{N \cdot r_{GI}[G(m,n), I]} \\
a_{I2} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{n-1}{N \cdot r_{GI}[G(m,n), I]} \\
a_{I3} &= \frac{1}{4\pi\epsilon_0} \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} g_{mn} \frac{N+2-m-n}{N \cdot r_{GI}[G(m,n), I]} \quad (3.63)
\end{aligned}$$

where:

$$\begin{aligned}
g_{mn} &= \Delta S_1 & (m \neq 1 \wedge m \neq N+1) \wedge (n=1) \\
g_{mn} &= \frac{1}{3}\Delta S_1 & (m=1 \vee m=N+1) \wedge (n=1) \\
g_{mn} &= \frac{1}{3}\Delta S_N & (m=1) \wedge (n=N+1) \\
g_{mn} &= \frac{1}{3}\Delta S_n + \frac{2}{3}\Delta S_{n-1} & [m=1 \wedge (n \neq N+1)] \vee [m=N+1 \wedge n \neq 1] \\
g_{mn} &= \Delta S_n + \Delta S_{n-1} & (m \neq 1 \wedge m \neq N+1) \wedge (n \neq 1 \wedge n \neq N+2-m)
\end{aligned}$$

Relations (3.62) and (3.63) are applicable for regular cases only, that is, in a case when an observed point I does not belong to any of the main nodes of the element. A singular case $r_{GI} = 0$ can be examined in the same way as before (explanations and relations (3.58)). In a case of square singular element, this element is examined as though it consists of two triangular elements resulting from a diagonal division.

In a similar way, a discrete form of the second order Fredholm integral equation is derived:

- for a square element

$$\sigma_J = \frac{2\sigma_{sJ}}{\epsilon_1 + \epsilon_2} + \frac{\lambda}{4\pi\epsilon_0} \sum_{e=1}^E \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} \frac{\vec{r}_{GJ}[G(u_n, v_m), J] \cdot \vec{n}_J^0}{r_{GJ}^3[G(u_n, v_m), J]} g_{mn} \cdot \sigma_e(m, n) \quad (3.64)$$

- for a triangular element

$$\sigma_J = \frac{2\sigma_{sJ}}{\varepsilon_1 + \varepsilon_2} + \frac{\lambda}{4\pi\varepsilon_0} \sum_{e=1}^E \sum_{m=1}^{N+1} \sum_{n=1}^{N+2-m} \frac{\vec{r}_{GJ}[G(m,n), J] \cdot \vec{n}_J^0}{r_{GJ}^3[G(m,n), J]} g_{mn} \cdot \sigma_e(m, n) \quad (3.65)$$

The occurrence of singularity in the relations (3.64) and (3.65) is solved in a way similar to the explanation for the curvilinear triangles.

3D FIELDS WITH AXIAL SYMMETRY

For the numerical solution of the integral equations which describe the electrostatic field in the systems with axial symmetry (rotationally - symmetric electrostatic fields), only the partial elements are used.

An axially-symmetrical element in the $r - z$ plane is projected onto the boundary lines, where the following are used:

- boundary line as a part of a straight line (projection of the cylinder shell, the projection of the cone shell, the projection of the cylinder basis, the projection of the torus sides of rectangular crosssection),
- boundary line as a part of an ellipse (projection of the ellipsoid, projection of the sphere or a part of the sphere, projection of the torus with circular crosssection).

These two types of partial lines include a great number of boundary surfaces. In a similar way, other sorts of boundary lines could be introduced as well; for example, boundary line as a part of a hyperbole, boundary line as a part of parabola etc.

We shall observe one arbitrary partial element, Fig. 3.14. For simplicity the straight boundary line is drawn. For the discretization of I Fredholm

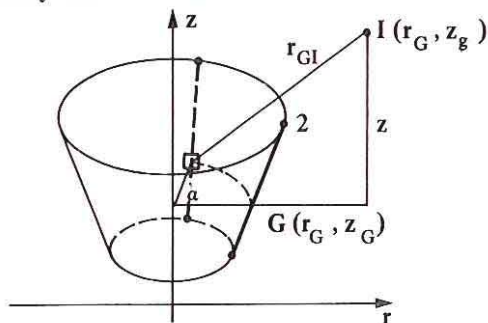


Figure 3.14: Arbitrary partial straight line element 1 - 2

equation, the potential in point I of the examined boundary elements is calculated, (nonessential indexes are omitted):

$$\Phi_{eI} = \frac{1}{4\pi\epsilon_0} \int_{S_e} \frac{\sigma(S)dS}{r_{GI}} = \frac{1}{4\pi\epsilon_0} \int_{S_e} \frac{\sigma(s)r_G d\alpha ds}{r_{GI}} \quad (3.66)$$

ds - boundary line element, $d\alpha$ - rotation angle element.

Let us introduce that $z = z_I - z_G$, so:

$$\begin{aligned} r_{GI}^2 &= [z^2 + (r_G + r_I)^2](1 - k^2 \sin^2 \beta) \\ \beta &= \frac{\pi - \alpha}{2} \\ k^2 &= \frac{4r_G r_I}{z^2 + (r_G + r_I)^2} \end{aligned} \quad (3.67)$$

Taking into consideration the axial symmetry, the calculation for one half of the surface S_e can be derived, so it follows that:

$$\alpha \in [0, \pi]; \quad \beta \in [0.5\pi, 0]; \quad d\alpha = -2d\beta \quad (3.68)$$

By substituting (3.67) and (3.68) in (3.66), and considering the given remark about the symmetry, it follows that:

$$\begin{aligned} \Phi_{eI} &= \frac{1}{\pi\epsilon_0} \int_s \int_0^{\pi/2} \frac{\sigma(s)r_G(s)ds}{\sqrt{z^2(s) + [r_G(s) + r_I]^2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} \\ \Phi_{eI} &= \frac{1}{\pi\epsilon_0} \int_s \frac{\sigma(s)r_G(s)K(k)}{\sqrt{z^2 + [r_G(s) + r_I]^2}} \cdot ds \end{aligned} \quad (3.69)$$

Introducing an appropriate local coordinate x for describing of the boundary line s , we can write:

$$ds = \sqrt{\left(\frac{dr_G}{dx}\right)^2 + \left(\frac{dz_G}{dx}\right)^2} dx = s(x)dx \quad (3.70)$$

$$\Phi_{eI} = \frac{1}{\pi\epsilon_0} \int_0^1 \frac{\sigma(x)r_G(x)s(x)K[k(x)]}{\sqrt{z^2(x) + [r_G(x) + r_I]^2}} \cdot dx \quad (3.71)$$

In the case of simplex basis function we have:

$$\sigma(x) = \sigma_1(1 - x) + \sigma_2 x \quad (3.72)$$

so:

$$\Phi_{eI} = a_{I1}\sigma_1 + a_{I2}\sigma_2 \quad (3.73)$$

$$\begin{aligned}
 a_{I1} &= \frac{1}{\pi \epsilon_0} \int_0^1 \frac{(1-x)r_G(x)s(x)K[k(x)]}{\sqrt{z^2(x) + [r_G(x) + r_I]^2}} \cdot dx \\
 a_{I2} &= \frac{1}{\pi \epsilon_0} \int_0^1 \frac{x r_G(x)s(x)K[k(x)]}{\sqrt{z^2(x) + [r_G(x) + r_I]^2}} \cdot dx
 \end{aligned} \quad (3.74)$$

For the element "straight line" we have:

$$\begin{aligned}
 r_G(x) &= (r_2 - r_1)x + r_1 = ax + r_1 \\
 z_G(x) &= (z_2 - z_1)x + z_1 = bx + z_1 \\
 s(x) &= \sqrt{a^2 + b^2}
 \end{aligned} \quad (3.75)$$

For the element "arc ellipse", it is convenient to use the central angle β as a parameter, and the local coordinate system defined with the ellipse centre coordinates (R_c, Z_c) and the ellipse axis; therefore we have:

$$\begin{aligned}
 \beta &= (\beta_2 - \beta_1)x + \beta_1; \quad r_G = R_c + A \cos \beta; \quad z_G = Z_c + B \sin \beta; \\
 s &= \sqrt{(A \sin \beta)^2 + (B \cos \beta)^2} |\beta_2 - \beta_1|
 \end{aligned} \quad (3.76)$$

The integrals (3.74) can be solved using a numerical procedure, where $s(x)$ is defined with 3.75 or 3.76. The difficulties appear only when the point I coincides with the node 1 or 2 of the observed element. That is, when we have singularity, the subintegral function in one point of integration has an infinite value. The solution for this problem will be discussed separately.

The electric field strength in the point J , caused by the differential part of the partial element (Fig. 3.15), is:

$$dE_e = \frac{dq}{4\pi\epsilon_0 r_{GJ}^2} = \frac{\sigma r_G d\alpha ds}{4\pi\epsilon_0 r_{GJ}^2} \quad (3.77)$$

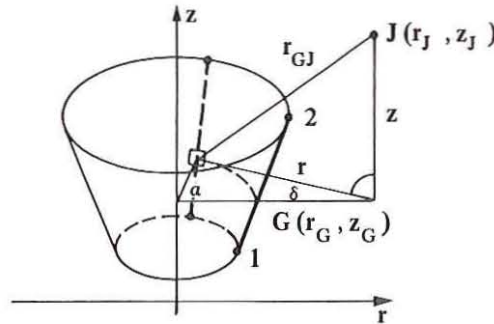


Figure 3.15: Arbitrary partial straight line element 1 - 2

The appropriate components of the electrical field are:

$$\begin{aligned} E_{er} &= \int \frac{r_d \cos \delta}{r_{GJ}} dE_e = \int \frac{r_J - r_G \cos \alpha}{r_{GJ}} dE_e \\ E_{er} &= \int_s \int_\alpha \frac{\sigma r_G}{4\pi\epsilon_0} \frac{r_J - r_G \cos \alpha}{r_{GJ}^2} d\alpha ds \end{aligned}$$

Using (3.67), (3.68) and (3.70), we can write:

$$\begin{aligned} E_{er} &= \int_s \left(\frac{\sigma r_G}{\pi\epsilon_0 [z^2 + (r_G + r_J)^2]^{3/2}} \int_0^{\pi/2} \frac{r_J + r_G \cos 2\beta}{(1 - k^2 \sin^2 \beta)^{3/2}} d\beta \right) ds \\ E_{er} &= \frac{1}{\pi\epsilon_0} \int_0^1 \frac{\sigma(x) r_G(x) s(x)}{2r_J \sqrt{z^2(x) + [r_G(x) + r_J]^2}} \cdot \\ &\quad \cdot \left(\frac{r_J^2 - r_G^2(x) - z^2(x)}{z^2(x) + [r_G(x) - r_J]^2} E(k) + K(k) \right) dx \quad (3.78) \end{aligned}$$

The z -component is calculated as follows:

$$\begin{aligned} E_{ez} &= \int \frac{z dE_e}{r_{GJ}} = \int_s \int_\alpha \frac{\sigma r_G z}{4\pi\epsilon_0 r_{GJ}^3} ds d\alpha \\ E_{ez} &= \int_s \left(\frac{\sigma r_G z}{\pi\epsilon_0 [z^2 + (r_G + r_J)^2]^{3/2}} \int_0^{\pi/2} \frac{1}{(1 - k^2 \sin^2 \beta)^{3/2}} d\beta \right) ds \\ E_{ez} &= \frac{1}{\pi\epsilon_0} \int_0^1 \frac{\sigma(x) r_G(x) s(x) z(x) E(k)}{(z^2(x) + [r_G(x) - r_J]^2) \sqrt{z^2(x) + [r_G(x) + r_J]^2}} dx \quad (3.79) \end{aligned}$$

If point J belongs to the boundary surface, and if the normal vector in the point J is given as $\vec{n}_J(n_{rJ}, n_{zJ})$, then the normal component of the electrical field strength in the point J is:

$$E_{enJ} = E_{er} n_{rJ} + E_{ez} n_{zJ} \quad (3.80)$$

In the case of the simplex base function, we have:

$$E_{enJ} = a'_{J1} \sigma_1 + a'_{J2} \sigma_2 \quad (3.81)$$

The relations for the calculation of a'_{J1} and a'_{J2} can be derived using (3.72), (3.78), (3.79) and (3.80), then (3.75) or (3.76) depending on the type of the partial element.

By applying (3.81) for each element, the discretization of the second Fredholm integral equation in the (3.48) form can be accomplished.

In the case that the point I coincides with the node 1 or 2 of the partial element, the integral (3.71) becomes singular (the point I is a singular

point). For solving of a singular integral, the point I needs to be isolated with a very small part, that is, the integral (3.74) should be split in two parts. For the singularity in the node 2 there will be:

$$a_{I1} = \int_0^{1-\delta} f_1(x)dx + \int_{1-\delta}^1 f_1(x)dx = a_{I1}^{(r2)} + \Delta a_{I1}^{(2)} \quad (3.82)$$

$$a_{I2} = \int_0^{1-\delta} f_2(x)dx + \int_{1-\delta}^1 f_2(x)dx = a_{I2}^{(r2)} + \Delta a_{I2}^{(2)} \quad (3.83)$$

or for the singularity in the node 1:

$$a_{I1} = \int_{\delta}^1 f_1(x)dx + \int_0^{\delta} f_1(x)dx = a_{I1}^{(r1)} + \Delta a_{I1}^{(1)} \quad (3.84)$$

$$a_{I2} = \int_{\delta}^1 f_2(x)dx + \int_0^{\delta} f_2(x)dx = a_{I2}^{(r1)} + \Delta a_{I2}^{(1)} \quad (3.85)$$

In the relations (3.82) to (3.85), first integral could be calculated in a regular way, taking for δ is a very small number, for example, $\delta = 0,001$.

Let the partial element be in the form of a "straight line" with the length s . Let us introduce a small length

$$s_0 = \delta \cdot s \quad (3.86)$$

with which we isolate a singular node. Then the contributions $\Delta a_{I1}^{(2)}$ and $\Delta a_{I2}^{(2)}$ could be calculated analytically according to the following relations:

$$\Delta a_{I1}^{(2)} = \frac{1}{\pi \varepsilon_0} \frac{\delta s_0}{2} \left[\left(\frac{1}{4} - \frac{a\delta}{9r_2} \right) - \left(\frac{1}{2} - \frac{a\delta}{3r_2} \right) \ln \frac{s_0}{8r_2} \right]; r_2 \neq 0 \quad (3.87)$$

$$\Delta a_{I1}^{(2)} = \frac{1}{\pi \varepsilon_0} \frac{\pi \delta^2 r_1}{4}; \quad r_2 = 0 \quad (3.88)$$

$$\begin{aligned} \Delta a_{I2}^{(2)} = & \frac{1}{\pi \varepsilon_0} \frac{s_0}{2} \left[\left(1 - \frac{\delta(a+r_2)}{4r_2} + \frac{a\delta^2}{9r_2} \right) - \right. \\ & \left. - \left(1 - \frac{\delta(a+r_2)}{2r_2} + \frac{a\delta^2}{3r_2} \right) \ln \frac{s_0}{8r_2} \right]; \quad r_2 \neq 0 \end{aligned} \quad (3.89)$$

$$\Delta a_{I2}^{(2)} = \frac{1}{\pi \varepsilon_0} \frac{\pi \delta r_1 (2 - \delta)}{4}; \quad r_2 = 0 \quad (3.90)$$

Replacing r_2 in (3.87), (3.88), (3.89) and (3.90) with r_1 , and r_1 with r_2 , that is, a with $-a$, index "2" with index "1" and index "1" with index "2",

we can get the relations for the calculation of the contribution $\Delta a_{I1}^{(1)}$ and $\Delta a_{I2}^{(1)}$:

$$\Delta a_{I1}^{(1)} = \frac{1}{\pi \epsilon_0} \frac{s_0}{2} \left[\left(1 - \frac{\delta(r_1 - a)}{4r_1} - \frac{a\delta^2}{9r_1} \right) - \left(1 - \frac{\delta(r_1 - a)}{2r_1} - \frac{a\delta^2}{3r_1} \right) \ln \frac{s_0}{8r_1} \right]; \quad r_1 \neq 0 \quad (3.91)$$

$$\Delta a_{I1}^{(1)} = \frac{1}{\pi \epsilon_0} \frac{\pi \delta r_2 (2 - \delta)}{4}; \quad r_1 = 0 \quad (3.92)$$

$$\Delta a_{I2}^{(1)} = \frac{1}{\pi \epsilon_0} \frac{\delta s_0}{2} \left[\left(\frac{1}{4} + \frac{a\delta}{9r_1} \right) - \left(\frac{1}{2} + \frac{a\delta}{3r_1} \right) \ln \frac{s_0}{8r_1} \right]; \quad r_1 \neq 0 \quad (3.93)$$

$$\Delta a_{I2}^{(1)} = \frac{1}{\pi \epsilon_0} \frac{\pi \delta^2 r_2}{4}; \quad r_1 = 0 \quad (3.94)$$

The derived relations for the calculation of the part of the coefficients a_{I1} and a_{I2} which corresponds to the singular part of the integral, could also be used for the partial element of the ellipse arc. In that case the length s_0 is a small length of the arc which corresponds to the central angle $\delta|\beta_2 - \beta_1|$ before the node "1" or the node "2", depending in which of these two nodes is the singular point. To make it simpler, the relation (3.86) can also be used for s_0 , where s is the ellipsis length which corresponds to the central angle $|\beta_2 - \beta_1|$ with the angle β_1 for the node "1".

When calculating elements a'_{J1} , a'_{J2} of the relations (3.78) to (3.81), we have no difficulties with the singularity, because, when $G = J$, only first integrals in (3.82) to (3.85) with the appropriate functions $f_1(x)$ and $f_2(x)$ need to be calculated.

3.2.4 Shape Optimization of the Boundary Surfaces

When planning modern power devices and apparatus, the main aim is to obtain an optimum shape of the electrode surfaces. There are various criteria for the optimum geometry. In electro-energetic apparatus, the most important criterion is the maximum operating security, which is frequently understood to mean avoidance of the possibility of critical stress on the conductor surfaces or over the interfaces between different dielectrics where a breakdown can happen.

The testing of the critical stresses over the boundary surfaces is a necessary stage in design of the electrical apparatus. The possibility of finding this stress by calculation, makes the whole process of experimental design and producing of the electrical apparatus much more efficient and cheaper.

To obtain the optimum shape of a boundary surface, very often the condition of equality of the tangential components of the field strength vector is used as optimization criterion. This condition is sufficient in all the cases where the breakdown only is possible. Recently, as a more common criterion, the equal value of the Maxwell stress over the dielectric surface is used.

If we need to investigate the optimum shape of some boundary surface in more complex problems, then it is necessary to use this second condition. Practical confirmation of this condition is reduced to the control of the values of the electrostatic pressure in the boundary surface with the following expressions [22]:

$$p = \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \left(E_t^2 + \frac{1}{\varepsilon_1 \varepsilon_2} D_n^2 \right) \quad (3.95)$$

$$p = \frac{1}{2} \frac{D_n^2}{\varepsilon_2} = \frac{1}{2} \varepsilon_2 \cdot E_{2n}^2 = \frac{1}{2} \varepsilon_2 \cdot E_2^2 \quad (3.96)$$

depending on what kind of the boundary interface is in question. First expression we use for the boundary interfaces between two dielectrics, while the second one is used for the interfaces which are boundaries between conductors and dielectrics.

Shape optimization of the boundary surfaces based on consecutive calculations with the introduction of the virtual surface

The investigation of the optimum shape of a boundary surface will be based here on consecutive calculations, with the criterion of reducing of field strength over the boundary surface. In the beginning, this can appear paradoxical. The application of this method on a little more complex problem would need a lot of time for the preparation and processing. In implementing this method on the computer, most of the time is consumed in forming the coefficient matrix $[A]$, equation 3.53. Therefore, it seems that all the methods which request the repetition of whole calculations, or a part of the calculation, are impractical. But it does not have to be so. Introducing virtual surfaces, the system which is being solved can be reduced to a small number of equations, that is, a small number of nodes. So, the whole procedure of consecutive calculations, until an optimum shape of a boundary surface is achieved, is reduced to a fast and economical process.

The idea that all other boundary surfaces are replaced by the virtual surface, except for the one which optimum shape is being calculated, makes this method a very applicable one. Thus, there is also a possibility of searching for the optimum shape in some wanted form which is easily produced.

The known procedures of optimization, except those which are based on consecutive calculations in a sense of the repetition of the whole calculation, do not have this option.

The procedure where the influence of one part of the complex system is replaced with the influence reduced to a virtual surface, will be illustrated in the following example.

Let us consider a complex system as shown on the Fig. 3.16. Let the boundary surface S_0 , whose optimum shape we want to find, have N_0 main nodes. We shall mark other surfaces $S_a, S_b, S_c, \dots, S_n$ with S_1 , and with N_1 main nodes on it. This means that when solving the full system, a coefficient matrix $[A]$ should be formed, with the dimensions $(N_0 + N_1, N_0 + N_1)$.

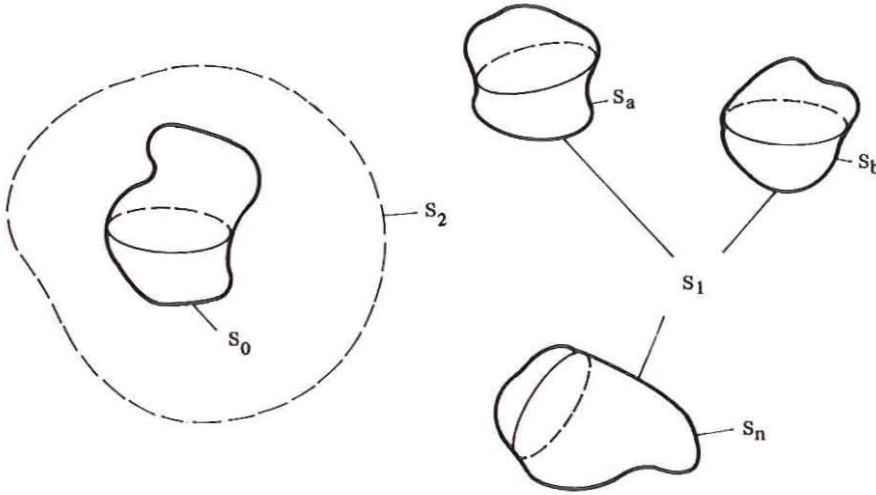


Figure 3.16: Arbitrary surface S_2 around the surface S_0 which is to be optimized

Around the S_0 surface, we close an arbitrary surface S_2 having N_2 main nodes after discretization. The matrix $[V_0]$ and $[V_1]$ are the known potentials on the surfaces S_0 and S_1 , respectively. Let the potentials in the main nodes N_2 of the virtual surface S_2 be the values of the matrix $[V_2]$.

Then, for the surface between S_1 and S_2 we can write:

$$[A_{11}][Q_1] + [A_{12}][Q_{21}] = [V_1] \quad (3.97)$$

$$[A_{21}][Q_1] + [A_{22}][Q_{21}] = [V_2] \quad (3.98)$$

while for the surface between S_0 and S_2 the following equations can be used:

$$[A_{00}][Q_0] + [A_{02}][Q_{20}] = [V_0] \quad (3.99)$$

$$[A_{20}][Q_0] + [A_{22}][Q_{20}] = [V_2] \quad (3.100)$$

In the previous equations, the matrix $[A_{11}]$ and $[A_{22}]$ and $[A_{00}]$ represent the parts of the coefficient matrix $[A]$ which are related to the nodes N_1 , N_2 and N_0 of the surfaces S_1 , S_2 and S_0 respectively, and they are the result of the boundary element charges on these surfaces. Thus, $[A_{11}]$ contains the coefficients which determine the contribution of the discrete charges in the node N_1 on each of the N_1 nodes in the surface S_1 . In short, these are the influences of the surface charges of the surface S_1 on its own nodes N_1 . However, the matrix $[A_{ij}]$, $i \neq j$, define the influence of the surface "i" on the nodes of the surface "j".

The matrix $[Q_{21}]$ represents unknown charges in the nodes N_2 which are the consequence of the influence of the discrete charges in the nodes of the surface S_1 . In the same way, $[Q_{20}]$ are the unknown charges in the nodes N_2 which are the result of influence of the discrete charges of the surface S_0 .

On the other side, the unknown potentials of the nodes N_2 of the virtual surface S_2 are determined from the known equation:

$$[A_{21}][Q_1] + [A_{20}][Q_0] = [V_2] \quad (3.101)$$

The equations (3.97) to (3.101) make a well defined equation system on the unknown $[Q_{21}]$, $[Q_{20}]$, $[Q_1]$, $[Q_0]$ and $[V_2]$. Our aim is to reduce this system of five equations to a system of two matrix equations with the unknown $[Q_0]$ and $[Q_{20}]$. By eliminating of the $[Q_{21}]$, $[Q_1]$ and $[V_2]$, we get the following equations system:

$$[A_{00}][Q_0] + [A_{02}][Q_{20}] = [V_0]$$

$$([A_{21}][A_{11}]^{-1} \cdot [A_{12}][A_{22}]^{-1}) [A_{20}][Q_0] + [A_{22}][Q_{20}] = [A_{21}][A_{11}]^{-1}[V_1] \quad (3.102)$$

that is:

$$\begin{aligned} [A_{00}][Q_0] + [A_{02}][Q_{20}] &= [V_0] \\ [A'_{20}][Q_0] + [A_{22}][Q_{20}] &= [V'_2] \end{aligned} \quad (3.103)$$

where:

$$[A'_{20}] = ([A_{21}][A_{11}]^{-1}[A_{12}][A_{22}]^{-1}) [A_{20}] = [B][A_{20}] \quad (3.104)$$

$$[V'_2] = [A_{21}][A_{11}]^{-1}[V_1]. \quad (3.105)$$

The (3.104) and (3.105) represent the reduced influence of the surfaces S_1 in the surface S_0 , but in the place of the virtual surface S_2 .

This equation system is the system whose matrix dimension is $(N_0 + N_2, N_0 + N_2)$, what is much less than the matrix dimension of the starting system.

From the electrostatics, it is known that through every equipotential surface, a conducting surface of the same potential can be put. Thus, the system of several electrodes can be reduced to a smaller number, and therefore can be easier solved. This principle is used in the previously presented procedure, but the virtual surface is treated as a discrete surface whose potentials at the nodes are known.

Reduced charges $[Q_{20}]$ and $[Q_{21}]$ of the secondary sources of the virtual surface can be treated as a simple charges layer, taking into account the continuity of the potential over the surface S_2 .

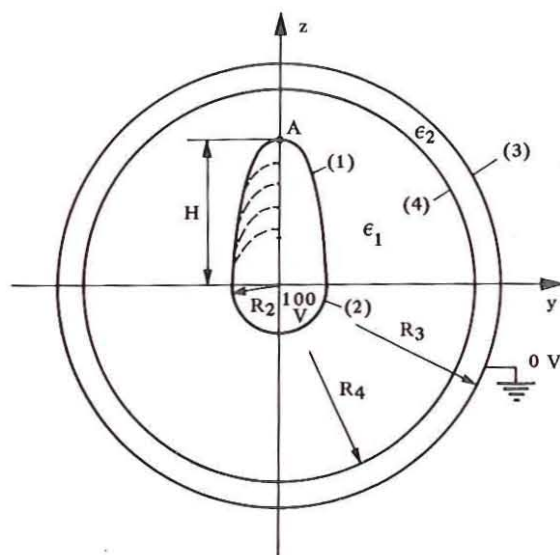


Figure 3.17: The system consisting of three electrodes which has to be optimized

Let us show, in an example, how the suggested procedure is used. Let the system shown on the Fig. 3.17 consists of one conductor described by the surfaces (1) and (2) with 100 V potential, one conductor (3) with 0 V potential, and two dielectrical media between them, with the permittivity ϵ_1 and ϵ_2 having as the common boundary the spherical surface (4). If the system of the electrodes is set up in this way, it is clear that the maximum

field value will be on the top of 100 V potential electrode, point A on the Fig. 3.17.

Let our aim be to find the optimum shape of electrode (1), such that the field in the point A is reduced for, let us say, 40%. In order to achieve this, it is necessary to reduce the height of paraboloid H . Let boundary surfaces (1) and (2), surface S_0 here, have N_0 main nodes. The spherical surfaces (3) and (4) are surfaces S_1 with N_1 nodes, whose influence in this system we want to reduce on the virtual surface S_2 , set as in the Fig. 3.18. Let the virtual surface S_2 be a spherical surface discretized so that it has N_2 nodes, while $N_2 \ll N_1$.

In the first step, let us solve the starting problem so that we calculate the field in the nodes 1, 4, 8, 12, 16, 20, 24 and 26 and the potential in N_2 nodes of the virtual spherical surface S_2 , Fig. 3.18. For solving this problem, we have to form the matrix of the coefficient $[A]$ with the dimensions $(N_0 + N_1, N_0 + N_1)$.

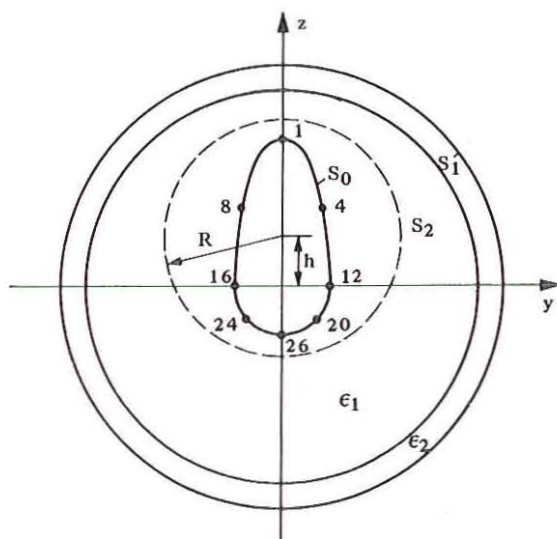


Figure 3.18: The arbitrary surface around the electrode which has to be optimized

In the second step, we observe the system with S_2 and S_1 boundary surfaces. The coefficient matrix for such a system is of the following dimensions: $(N_1 + N_2, N_1 + N_2)$. After dividing this matrix, according to the expression (3.104), we obtain the needed matrices $[A_{11}]$, $[A_{12}]$, $[A_{21}]$ and $[A_{22}]$. The dimensions of these matrices are respectively $(N_1 \times N_1)$,

$(N_1 \times N_2)$, $(N_2 \times N_1)$, $(N_2 \times N_2)$. Thus, we form the following matrices:

$$[B] = [A_{21}][A_{11}]^{-1}[A_{12}][A_{22}]^{-1} \quad (3.106)$$

$$[V_2'] = [A_{21}][A_{11}]^{-1}[V_1] \quad (3.107)$$

The dimensions of these matrices are $(N_2 \times N_2)$ and $(N_2 \times 1)$.

In the third step, we observe the system where the boundary surfaces S_0 and S_2 are with the total number of nodes $N_0 + N_2$. In accordance with expressions (3.103) - for this kind of system there need to be defined the matrices $[A_{00}]$, $[A_{02}]$ and $[A_{20}]$. In this way we have $[A_{02}]$ and $[A_{20}]$. In this way we have completely defined all the necessary matrices for the equations system (3.103), solving which we get the unknown charges distribution $[Q_0]$ and $[Q_{20}]$ which serves as a basis for finding the potential in the same points as in the first step.

This results in equivalent values for charges in the nodes N_0 , equivalent field values in the same points, and the equivalent potential values in nodes N_2 which are derived in the first step, without using the virtual surface.

By changing the paraboloid height, only matrices $[A_{00}]$, $[A_{02}]$ and $[A_{20}]$ are changed, so by their forming again, we can find a new charges distribution, that is, new field values in the points which are of interest for us.

Fig. 3.17 and 3.18							
$R_2 = 2, R_3 = 9, R_4 = 8, R = 5, h = 2$							
	$H = 6$		$H = 5$	$H = 4$	$H = 3$	$H = 2$	
	a	b	b	b	b	a	b
$E_n(1)$	133.593	133.592	123.265	111.455	96.723	78.641	78.375
$E_n(26)$	57.119	57.085	57.298	57.598	58.011	58.629	58.583
$Q_0(1)$	7.254	7.151	7.353	7.231	6.752	5.726	5.707
$Q_0(2)$	3.527	3.508	3.653	3.837	4.075	4.346	4.332
$Q_0(10)$	4.069	4.054	4.231	4.487	4.882	5.568	5.552
$Q_0(18)$	4.732	4.707	4.742	4.778	4.809	4.835	4.812
$Q_0(26)$	4.222	4.217	4.232	4.255	4.289	4.344	4.334
(a) - Calculated without the virtual surface							
(b) - Calculated with the virtual surface							

Table 3.1: The results of the electrostatic field calculation (points 1, 26) and charges (1, 2, 10, 18, 26) during the optimization procedure

For the example mentioned, if the discretization is performed so that

$N_1 = 52$; $N_2 = 26$; $N_0 = 26$, with the geometrical parameters: $R_2 = 2$; $R_3 = 9$; $R_4 = 8$; $R = 5$; $h = 2$ and $H = 6, 5, 4, 3, 2$, and the dielectrical constants $\epsilon_{r1} = 1$ and $\epsilon_{r2} = 2$, then we get the results as shown in the table 3.1.

In the rows marked with "a" in the table 3.1, calculated values are given if the starting system is to be solved directly, that is, without introducing the virtual surface. It is readily seen that the calculation in which the system with the virtual surface is solved, gives practically identical results.

Automatic Shape Optimization of the Boundary Surfaces

Let us assume that we want to optimize the boundary interface between two dielectrical surfaces with the constants ϵ_1 and ϵ_2 (Fig. 3.19). After the calculation of the electrical stress in all points (according the relation (3.95)) on the boundary surface which is the object of the optimization, we define the "referent value" of Maxwell's pressure. The main aim of optimization is the reductions of the highest value of the electrical stress under the "referent value". The criterion stated in this way, is called the "local" optimization criterion.

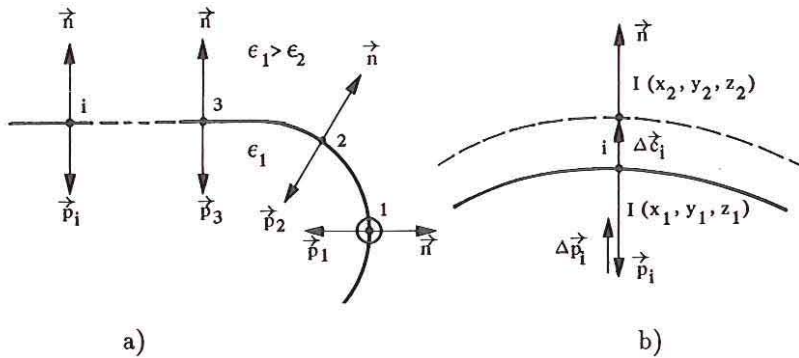


Figure 3.19:

- a) The direction of the pressure vector and normal vector $\vec{n}$ along the boundary interface between different media
- b) Virtual moving of the surface

Let us assume that $\epsilon_1 > \epsilon_2$. Then:

$$\Delta p_i = |p_i - p_{ref}| \quad (3.108)$$

Vector $\Delta \vec{p}_i$ is in the direction of the normal vector $\vec{n}_i$ (Fig. 3.19b), which

means that it reduces the absolute value of vector $\vec{p}_i$.

$$\Delta \vec{p}_i = \Delta p_i \cdot \vec{n}^0(I) = \Delta p_x \vec{i} + \Delta p_y \vec{j} + \Delta p_z \vec{k} \quad (3.109)$$

The correction vector $\Delta \vec{c}_i$ is then:

$$\Delta \vec{c}_i = \Delta p_x c_i \vec{i} + \Delta p_y c_i \vec{j} + \Delta p_z c_i \vec{k} \quad (3.110)$$

where c_i is the correction factor.

Taking into consideration that vector $\Delta \vec{c}_i$ is defined as:

$$\Delta \vec{c}_i = (x_2 - x_1) \vec{i} + (y_2 - y_1) \vec{j} + (z_2 - z_1) \vec{k} \quad (3.111)$$

relations (3.110) and (3.111) give:

$$x_2 = x_1 - \Delta p_x c_i \quad y_2 = y_1 - \Delta p_y c_i \quad z_2 = z_1 - \Delta p_z c_i \quad (3.112)$$

where (x_2, y_2, z_2) are the new coordinates.

For $\varepsilon_2 < \varepsilon_1$, in a similar way we get:

$$x_2 = x_1 + \Delta p_x c_i \quad y_2 = y_1 + \Delta p_y c_i \quad z_2 = z_1 + \Delta p_z c_i \quad (3.113)$$

Moving the boundary surface in this way, step by step, we decrease the greatest electrical stress until its value falls under allowed value (referent value Δ).

$$\left| \frac{p_i - p_{ref}}{p_{ref}} \right| \leq \Delta \quad (3.114)$$

This "local criterion" of moving of the points, often gives unwanted oscillation effect. In these cases, it is beneficial to introduce a stricter criterion which allows the increase of the electrical stress in the points which are not critical, for the reduction of the electrical stress in the critical points.

Local optimization criteria is applicable when optimizing a system of a symmetrical structure or separately distinguished body, where a decrease of electrostatic pressure in each point of optimization is assumed. In asymmetrical structures, the use of local criteria leads to the oscillation of the optimization procedure.

To overcome this difficulty, we have introduced a stronger criterion, so called "master" criterion. This criterion constrains the moving of the surface in such a direction which permits an increase in the electrostatic pressure in some points of the surface, in order to decrease the pressure in the critical points. To explain this, let us assume that we have a distribution of the pressure as shown on Fig. 3.20, line *a*.

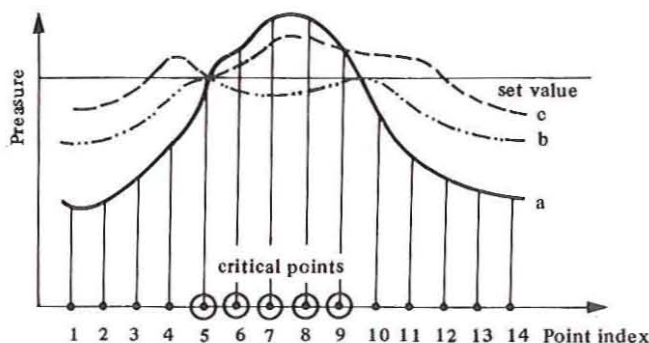


Figure 3.20: The distribution of the pressure values during the optimization iterations

The first step in this procedure is the "trial step" according to which the decision is made about the direction where to move the surface in order to reduce the pressure in the critical point (or points). Let us examine three cases:

- following the trial step, the pressures in all points are decreased,
- following the trial step, the pressures in all points are increasing,
- in some points the pressure is decreased, whereas in others it is increasing.

It is clear that the application of the "local criteria" is possible in the first case only, while in the other two, oscillations would appear.

Applying the "master" criterion we continue moving of the surface in some points in the direction which provides the increase of the pressure in those points in order to decrease the pressure in the critical points.

The procedure is repeated until the pressures in the critical points achieve the values smaller than those assigned in advance, line *b* on Fig. 3.20, or until the pressure in other points reaches critical value, line *c* on Fig. 3.20.

Chapter 4

Software Description

4.1 ELFI

Geometry

ELFI calculates electrostatic field in two dimensional planar (2D) and axisymmetric (AS) geometries. The geometry can be defined interactively using the graphic modeler or via an input file. Three segment types can be applied for modeling:

- arc
- line
- single contour points

All these segments are divided into two categories: electrode boundaries and boundaries between two dielectrics.

The geometry of the calculated arrangement can be significantly reduced by using the symmetry on a plane as well as electric images on the following conducting surfaces:

- plane (2D,AS)
- cylinder (2D)
- sphere (AS)

ELFI is capable of generating and solving:

- open region problems

- simplified arrangement models with not fully defined geometry (open contours)
- problems with very thin conducting foils

Method

ELFI uses the charge simulation with discrete charges lying outside the boundaries. The discretization is associated with generating collocation points on boundaries and simulating one or two charges for each point. Four charge types can be used:

- ring charges (AS)
- line charges on the rotation axis (AS)
- point charges on the rotation axis (AS)
- infinite line charges (2D)

The charge simulation can be surface-oriented or region-oriented. The user can choose the solution method by setting the appropriate solver configuration switch.

Boundary Conditions and Material Properties

The boundary conditions are assigned to all collocation points on electrode boundaries. The set of available boundary conditions includes:

- constant potential
- floating potential
- distributed potential
- normal field strength

ELFI can calculate arrangements with many different lossless dielectric materials.

Presentation of Results

Results in form of field vectors on electrodes and dielectric boundaries are automatically plotted on the screen after the solver has finished. If the results are acceptable, the calculations can be continued for points lying outside the boundaries. In a selected display area, the user can interactively calculate:

- Field vectors in points picked on the screen
- Equipotential lines for selected potentials
- Equifield lines for selected field strengths
- Field lines

Other post-processing features include graphs ($X - Y$ -plots) along any contour segment(s) as well as capacities of the complex multielectrode and multidielectric arrangements.

Control of Solution Quality

The accuracy of results can be checked by means of:

- potential deviation in test points on electrode boundaries
- field vectors angle deviation on electrodes
- deviation of tangential field strength on dielectric boundaries

These quantities are automatically calculated for the whole arrangement model and are displayed immediately after each solver job. They can also be evaluated locally in selected regions.

Software Structure

- Integration: ELFI provides a fully integrated free-standing tool for electrostatic field calculations. The pre-processing, solver and post-processing are integrated into one program. The ELFI modules correspond to user activities which can be performed within an interactive session:
 - modeling
 - discretization
 - solving (calculating charge values)
 - calculations of field strengths and potentials on boundaries
 - calculations of equipotential, equifield and field lines
 - presentation of results

The order in which the modules are activated can be selected by user.

- Data base: All input data and results are available on a data base. The memory management within the data base is dynamic so that the current size of the allocated memory corresponds always to the size of the solved problem.
- Open architecture: The modular and object-oriented structure of ELFI allows an easy integration of new or enhanced solution sequences. The extension to 3D geometries is possible.

Other Features

- written in C
- batch option available
- all output in ASCII-format for easy integration with other software and data exchange between different computers
- plotting of screen images

4.2 CSMDDD

Geometry

CSMDDD is a program for the calculation of three dimensional electric field arrangements consisting of components with axial symmetry. The geometry must be described by a set of data for each single axial symmetric component. These components are defined by straight lines and arcs in a local coordinate system. After the definition of the geometry in this local system, the components are shifted to the actual place in a global coordinate system. A graphical output for the check of the input data is available (CalComp HCBS routines).

The distance between the collocation points can be controlled by the input of a "density factor".

Imaging in order to consider an infinite grounded plane is possible.

Method

CSMDDD is based on surface-oriented CSM using discrete charges with partially constant charge density (ring segment charges and straight line segment charges). The position of the simulation charges is calculated automatically.

Boundary Conditions and Material Properties

CSMDDD is limited to electric field problems with Dirichlet boundary conditions.

Only two different lossless dielectrics can be introduced.

Results

Default option is the output of the field strength components in the collocation points on dielectric boundaries.

Optionally the potential and the field strength components in preselected points on the electrode surfaces, dielectric boundaries and within the dielectrics can be calculated.

The graphic output is limited to the presentation of vector arrows in different preselected plane sections through the 3D geometry.

Control of Solution Quality

Several options enable the accuracy to be assessed, as is usual for CSM programs:

- potential deviation between collocation points on electrode boundaries
- field vectors angle deviation on electrode boundaries
- deviation of tangential field strength on dielectric-boundaries.

Structure

CSMDDD is written in FORTRAN.

It is structured in a pre-processor, solver and post-processor. The solver uses a Gauss elimination method for the solution of the equation system with a full asymmetric matrix. The solution process can be controlled by the selection of different pivot element strategies.

Remarks

CSMDDD was developed more than 15 years ago. It was used during the cooperation for comparison purposes.

4.3 EFCA_2D

Geometry

EFCA_2D calculates electrostatic field with axisymmetric geometry. The geometry can be defined interactively using the graphic modeler or via an input file. Two types of boundary lines can be applied for modeling:

- arc ellipse
- straight line

All these segments are divided into two categories: electrode boundaries and boundaries between two dielectrics.

The geometry of the calculated arrangement can be significantly reduced by using the symmetry on a plane as well as electric image with respect to the ground.

With EFCA_2D it is possible to generate and solve:

- open region problems
- simplified arrangement models with not fully defined geometry (open contours)
- problems with very thin conducting foils
- problems with the electrodes being on the floating potentials, surrounded by the different types of dielectric.

Method

EFCA_2D uses the the boundary element method with two different types of the partial elements: straight line and elliptic arc segments.

Boundary Conditions and Material Properties

The boundary conditions are assigned to all electrodes on the known potential. It is also possible to take into account the bound surface charge situated on the border between two dielectrics.

EFCA_2D can calculate arrangements with different lossless dielectric materials, with an unlimited number of the dielectrics.

Presentation of Results

The calculation results are given in the protocol or can be graphically presented. In a selected display area, the user can calculate and observe:

- The components of the field strength vector or field intensity in the boundary nodes. The node with maximum field strength is separately marked.
- Equipotential lines for selected potentials

The calculation results can be presented in the different windows, using the data basis which is formed as a result of the solver running.

Other post-processing features include the calculation of the capacities of the complex multielectrode and multidielectric arrangements.

Software Structure

EFCA-2D provides a fully integrated free-standing tool for electrostatic field calculations. The pre-processing, solver and post-processing are integrated into one program. The user can directly, or from the environment, activate the following procedures:

- the discretization of the model
- pre-processing, checking of the input data and their interactive correction
- main solver (first run)
- auxiliary solver (second run), field and potential calculation in the prescribed points
- post-processing.

The modular and object-oriented structure of EFCA-2D allows an easy integration of new or enhanced solution sequences.

Other Features

- written in C, FORTRAN, PASCAL
- batch option available
- printing and plotting of screen images.

4.4 EFCA_3D

The package EFCA_3D consists of two separate BEM programs, TEFCA and POLOPT [26]. POLOPT uses curvilinear triangle elements for modeling, while TEFCA also uses a "typical elements" as a parts of real surface.

Geometry

EFCA_3D calculates electrostatic field in the arbitrary 3D geometry. The "model builder" session is performed using separate software package FAM¹ or over an input data file. The modeling of the boundary surfaces can be performed by

- curvilinear triangle elements with quadratic approximation
- partial elements, being the parts of "typical" boundary surfaces of first and second order

The plane symmetry of the model, as well as a "electrical imaging" facility is also taken into consideration.

EFCA_3D permits the modeling and calculation of the following problems:

- open region problems
- problems with thin conducting foils
- problems with floating potentials, surrounded by the different types of dielectric.

Method

EFCA_3D uses the the boundary element method with two different types of the partial elements: curvilinear triangle elements with quadratic approximation and partial elements.

Boundary Conditions and Material Properties

The boundary conditions are implicitly assigned to all electrodes on the known potential, i.e. during modeling session it is not necessary to prescribe separately the boundary conditions as by Finite Element Method.

EFCA_3D can calculate arrangements with different lossless dielectric materials, with an unlimited number of the dielectrics.

¹FAM is graphic pre-processing and post-processing package from FEES Ltd Oakington Cambridge CB4 5BA England.

Presentation of Results

EFCA_3D has very powerful possibilities of the result presentation using FAM facilities. It possesses all features of state-of-art graphic pre-processing and post-processing packages.

Using one of the available options (shaded element, contouring, vector presentation etc.) it is possible to observe:

- charge distribution over arbitrary chosen geometry
- potential distribution over arbitrary chosen geometry
- field strength distribution (scalar or vector) over arbitrary chosen geometry
- charge, potential and field vector presentation in arbitrary intersection plane in 3D space
- distribution of the charge, potential or field values over selected line in the graph form.

Software Structure

EFCA_3D provides a fully integrated free-standing tool for electrostatic field calculations. The pre-processing, solver and post-processing are integrated into one program. The user can directly, or from the environment, activate the following procedures:

- the discretization of the model
- pre-processing
- solver
- post-processing.

The modular and object-oriented structure of EFCA_3D allows an easy integration of new or enhanced solution sequences.

Other Features

- written in FORTRAN
- batch option available
- printing of screen images.

surface sections $A-B-C$ and $D-E$ (Fig. 5.2) were selected as parts with the maximum electric stress. The comparison of the field strength values as a function of the distance s (Fig. 5.2) in Table 5.2 shows that the deviation between the results calculated with BEM and CSM is less than 0.5% in the range with the highest field strength. It is a maximum in the points B' and D with 1% and 1.3% respectively.

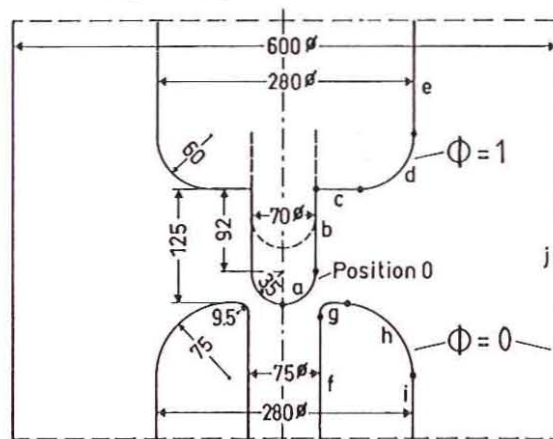


Figure 5.1: Example I: Disconnecter

Part (Fig. 5.1)	Number and type of	
	boundary elements	simulation charges
a	14	14 ring 1 point
b	16	15 ring
c	6	6 ring
d	5	5 ring
e	9	9 ring
f	16	15 ring
g	24	24 ring
h	10	11 ring
i	3	2 ring
j	15	14 ring
Σ	118	116

Table 5.1: Example I: Number of boundary elements and simulation charges

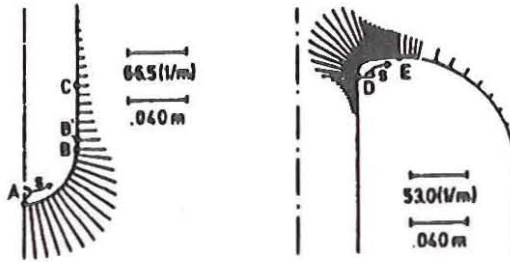


Figure 5.2: Example I: Field strength distribution on the surface

	s [mm]	E' [1/m]		Deviation [%]
		BEM	CSM	
A	0.00	58.369	58.307	< 0.5
	7.85	59.536	59.586	< 0.5
	15.71	63.075	63.081	< 0.5
	23.56	66.501	63.387	< 0.5
	31.42	64.115	63.979	< 0.5
	39.27	54.830	54.807	< 0.5
	47.12	43.724	43.744	< 0.5
B	54.98	32.017	32.259	- 0.8
B'	60.73	25.660	25.916	- 1.0
	72.23	19.661	19.705	< 0.5
	83.73	15.662	15.674	< 0.5
C	95.23	12.494	12.492	< 0.5
D	0.00	28.972	29.350	- 1.3
	2.46	40.634	40.599	< 0.5
	4.91	48.667	48.602	< 0.5
	7.37	52.816	52.734	< 0.5
	9.82	51.940	51.844	< 0.5
	12.28	43.947	44.223	- 0.6
	14.37	37.115	37.425	- 0.8
	18.54	30.887	30.936	< 0.5
	22.72	26.459	26.486	< 0.5
	26.89	23.042	23.069	< 0.5
	31.07	20.321	20.330	< 0.5
E	33.16	19.148	19.180	< 0.5

Table 5.2: Example I: Comparison of the calculated field strength values E' (1/m) on the surface between A-B, A-C and D-E (Fig. 5.2)

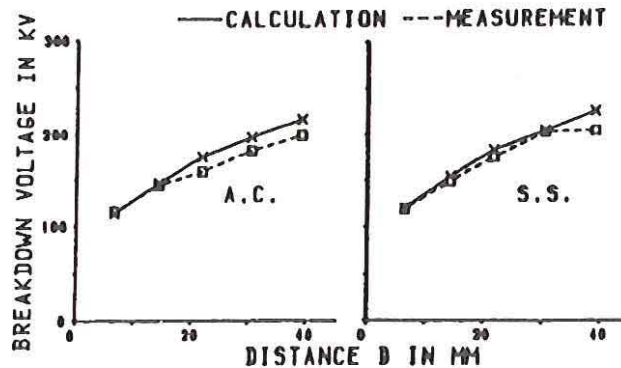


Figure 5.3: Example I: Measured breakdown voltages for switching surge 200/2500 (S.S.) and a.c. 50 Hz in comparison with calculated values

For the disconnector arrangement experiments in SF_6 with switching surge 200/2500 (S.S.) and a.c. 50 Hz were made in addition. The breakdown voltage was measured for 5 different

breakdown voltage was evaluated from 30 single measurements for each position of the contact. The results are plotted in Fig. 5.3 together with the calculated values, derived from field calculation using a critical field strength of 97.5 kV/cm for a.c. and 102.3 kV/cm for S.S. regarding the curvature radius $r = 3.5$ cm of the moving contact and the different wave-shapes [23]. The maximum differences between the calculated values and the mean values of the measured breakdown voltages is less than 5% for gap distances less than 25 mm and in the range of 10% for larger gap distances.

5.2 Example II. Insulator

Example II (Fig. 5.4) is an insulator arrangement similar to an example already used for a comparison of different methods [24]. In the same way as for example I Table 5.3 shows the way of discretization and Fig. 5.5 the field strength distribution on the boundaries with the maximum stress. The comparison of the field strength values (Table 5.4) was restricted to the surface part A-B-C of the upper protection electrode, where the field strength is a maximum. The maximum difference between the calculated values amounts to 0.7% in the area where the curvature of the electrode surface changes.

Part (Fig. 5.4)	Number and type of	
	boundary elements	simulation charges
<i>a</i>	10	9 ring 1 point
<i>b</i>	10	10 straight line
<i>c</i>	11	10 ring
<i>d</i>	10	11 ring
<i>e</i>	48	41 double ring 8 single ring
<i>f</i>	10	11 ring
<i>g</i>	10	10 ring
<i>h</i>	10	12 straight line
<i>i</i>	26	22 ring
Σ	145	145

Table 5.3: Example II: Number of boundary elements and simulation charges

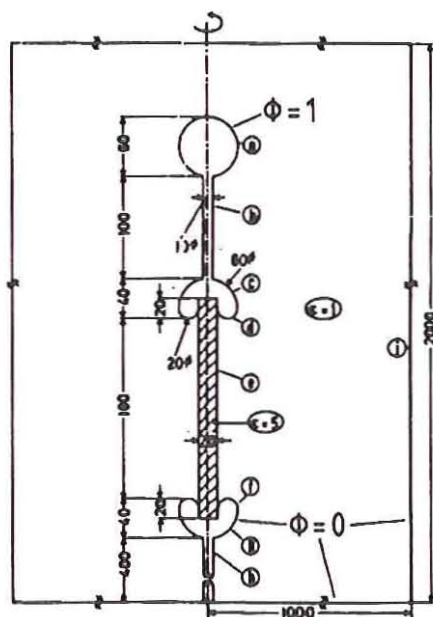


Figure 5.4: Example II: Insulator arrangement

	s [mm]	E' [1/m]		Deviation [%]
		BEM	CSM	
A	0.00	31.963	32.011	< 0.5
	3.14	36.955	36.872	< 0.5
	6.28	40.364	40.262	< 0.5
	9.42	41.940	41.735	< 0.5
	12.57	41.748	41.443	+ 0.7
B	15.71	37.081	37.112	< 0.5
	19.07	32.392	32.624	- 0.7
	22.44	30.830	30.857	< 0.5
	25.81	29.409	29.419	< 0.5
	29.17	28.152	28.155	< 0.5
	32.54	26.885	26.876	< 0.5
	35.90	25.538	25.516	< 0.5
C	39.27	24.027	23.996	< 0.5

Table 5.4: Example II: Comparison of the calculated field strength values E' (1/m) on the surface between A-B-C (Fig. 5.5)

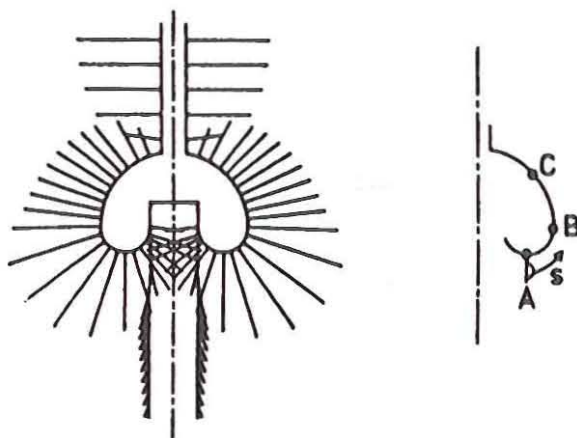
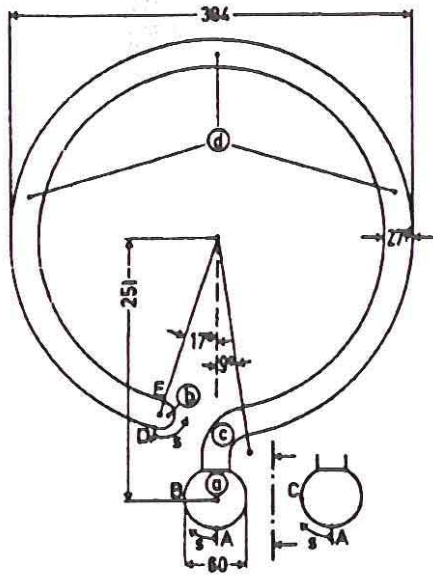


Figure 5.5: Example II: Field strength distribution on the upper electrode



Part (Fig. 5.6)	Number and type of	
	boundary elements	simulation charges
<i>a</i>	56 spherical	57 ring segment
<i>b</i>	32 spherical	33 ring segment
<i>c</i>	8 cylindrical	32 ring segment
	24 toroidal	
<i>d</i>	144 toroidal	142 ring segment
Σ	264	264

Table 5.5: Example III: Number of boundary elements and simulation charges

The comparison of the field strength values along the paths $A-B$, $A-C$ and $D-E$ shown in Fig. 5.6 is listed in Table 5.6. The deviations between the results calculated by means of the two methods in this 3D case are higher in comparison with the 2D fields, but they are below 3%.

	s [mm]	E' [1/m]		Deviation [%]
		BEM	CSM	
A	0.00	22.420	22.790	- 1.6
	11.46	22.850	22.540	+ 1.4
	22.93	21.880	21.690	+ 0.9
	34.35	20.450	20.190	- 1.3
B	45.85	18.180	17.940	+ 1.3
A	0.00	22.420	22.790	- 1.6
	11.46	22.910	22.590	+ 1.4
	22.93	22.130	21.920	+ 1.0
	34.35	21.030	20.730	+ 1.4
C	45.85	19.220	18.920	+ 1.6
D	0.00	16.700	16.690	< 0.5
	10.60	15.790	15.650	+ 0.9
	21.21	13.560	13.860	- 2.2
	31.81	16.630	16.430	+ 1.2
E	42.41	14.640	14.880	- 1.6

Table 5.6: Example III: Comparison of the calculated field strength values E' (1/m) on the surface between $A-B$, $A-C$ and $D-E$ (Fig. 5.6)

5.4 Example IV. 420 kV Transformer

Example IV is a calculation of the electrostatic field of a 420 kV transformer with the shielding electrode being on the floating potential.

Having in mind that in this example the degree of the axial asymmetry is relatively small, the same example is solved once as a true 3D problem, and once as an approximated axially symmetrical problem.

The discretization mesh is shown in Fig. 5.7

The results are presented using field vector presentation, Fig. 5.8 where the length of the vector represents a field intensity.

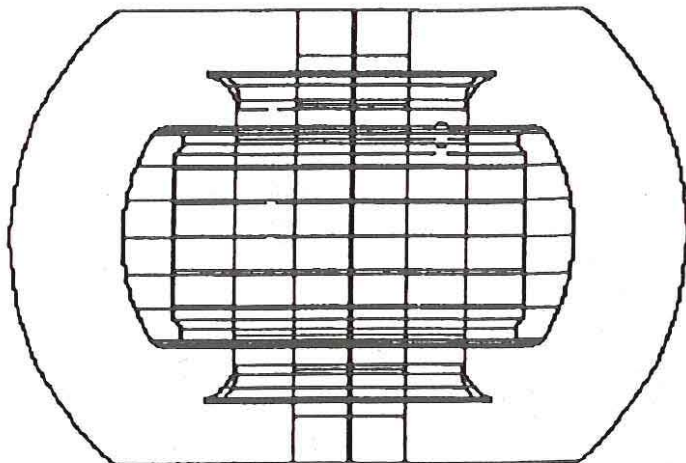


Figure 5.7: Example IV: 420 kV transformer, grid of boundary elements

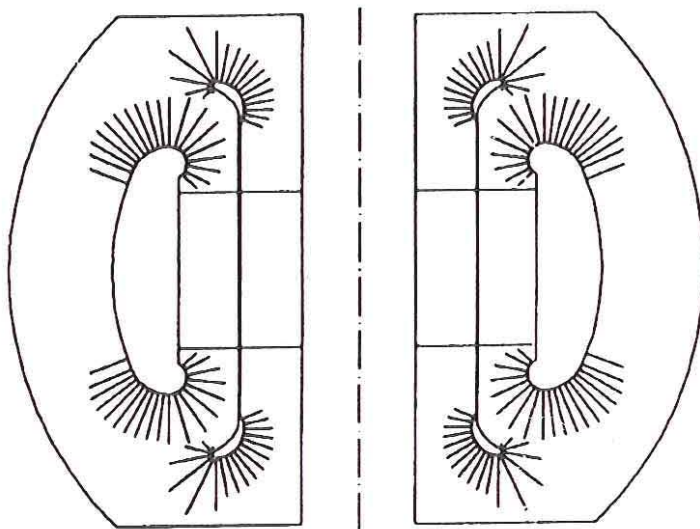


Figure 5.8: Example IV: Field strength distribution on a part of transformer

5.5 Example V. 420 kV Disconnecter

Example V is a 420 kV disconnector in with SF_6 gas insulated equipment.

The problem is solved using developed software package POLOPT.

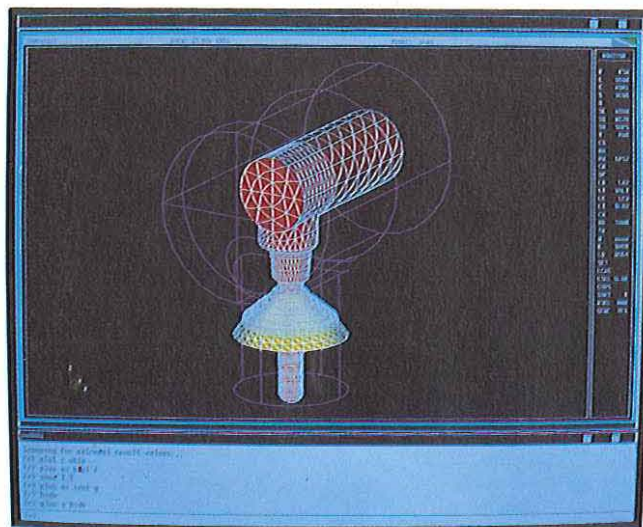


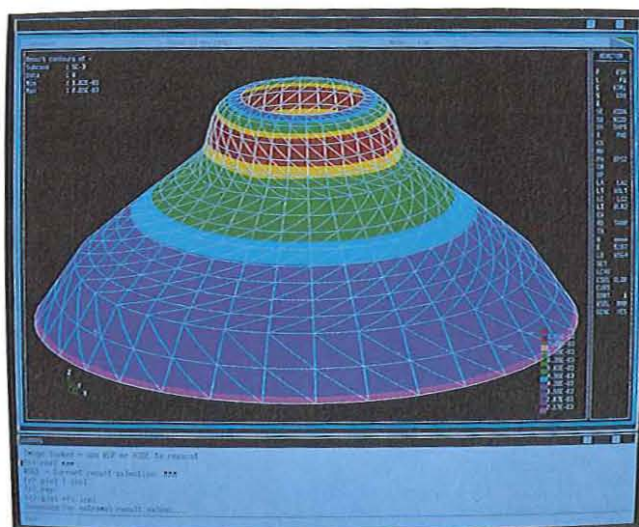
Figure 5.9: Example V: Disconnecter with "shad" insulator

In order to analyze the influence of the different shape of the insulator two arrangements were solved: disconnector with "flat" insulator and disconnector with "shed" insulator, Fig. 5.9. The number of the nodes and elements for these problems were:

	elements	nodes
problem with "flat" insulator	5187	10340
problem with "shed" insulator	9859	19684

The most interesting is the field distribution over insulator. Figures 5.10 and 5.11 show a field distribution over "flat" and "shed" insulator, respectively, represented by shaded elements.

Figures 5.12 and 5.13 present the field distribution along the top and bottom boundary line of "flat" and "shad" insulator, respectively.



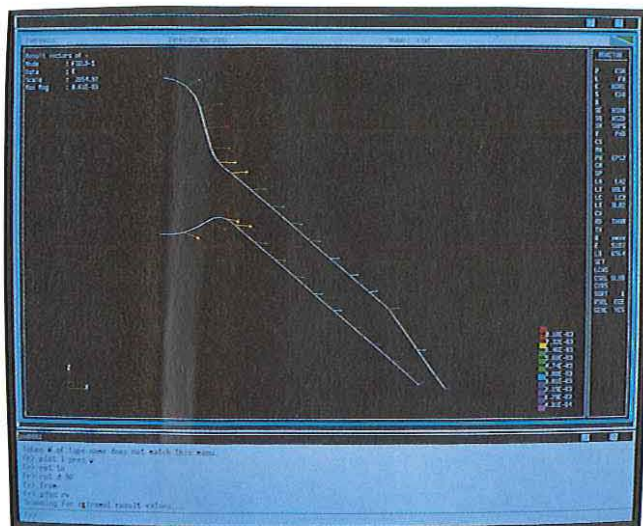


Figure 5.12: Example V: Field distribution along the top and bottom boundary line of "flat" insulator

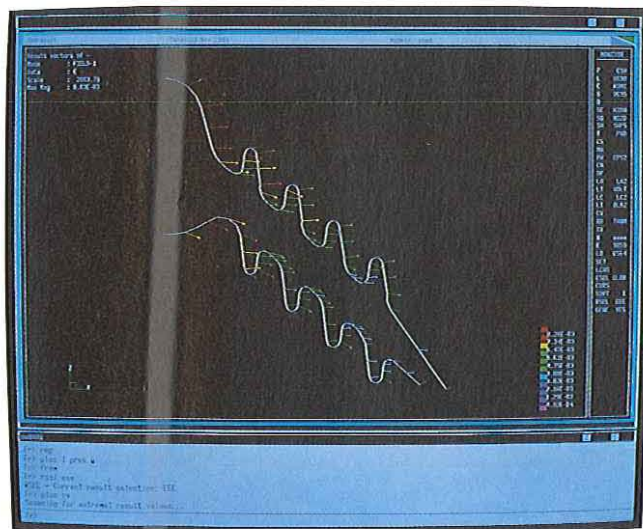


Figure 5.13: Example V: Field distribution along the top and bottom boundary line of "shad" insulator

5.6 Example VI. HV Bushing

The classic (surface-oriented) charge simulation method [1] requires that discrete charges assigned to an electrode surface be positioned within the electrode. This requirement usually leads to difficulties when modelling very thin electrodes like conducting foils. In the model of a foil shown in Fig. 5.14a, two layers of charges separated from each surface by a distance x have been simulated. This distance should be approximately 1.0 to 2.0 times greater than the distance y between adjacent collocation points (see section 3.1.4). When using the surface-oriented method, the right and the left layer must not overlap, as in Fig 5.14b. Consequently, the minimum thickness of the foil can only be reduced by increasing the number of simulated charges which results in a high dimension of the equation system.

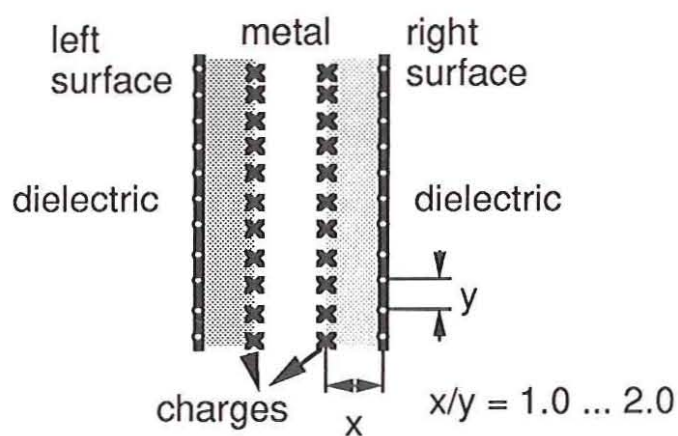
When using the region-oriented method the foil is regarded from the viewpoint of an observer associated with a region. The observer can only see the boundaries of his region with charges assigned to them. Charges assigned to other regions are not of interest for him, since they are not used directly for field calculation in his region. In Fig. 5.14c the observer situated within region R_i can only see the left surface of the foil with appropriate charges lying outside his region. The simulation charges that lie coincidentally within his region are associated with another surface, and therefore unknown to him. From such a viewpoint the thickness of the foil is not important – it can even equal zero.

The foils with zero thickness have been applied in the model of a HV bushing shown in Fig. 5.15. One of the 5 cylindrical foils is at zero potential, the other 4 are at floating potentials. In order to create different regions on both sides of each foil, the fictitious dielectric boundaries have been introduced (dashed lines in Fig. 5.15). The arrangement has been divided into 7 regions.

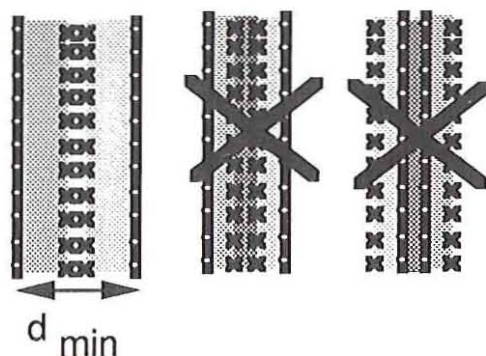
Charges simulated for the whole arrangement produce a rather unreadable image shown in Fig. 5.16a. However, this image is clear when regarding regions separately – each region is surrounded by a set of charges like in Fig. 5.16b. No charges have been positioned directly on the foils. There is no free charge which can be assigned to a foil. Therefore, the additional equations to obtain the floating potentials have been formulated by integrating the flux over the both surfaces of each foil.

Results in form of equipotential lines obtained with the program ELFI are shown in Fig. 5.17.

a) Model of a foil:



b) Surface – oriented charge simulation:



c) Region – oriented charge simulation:

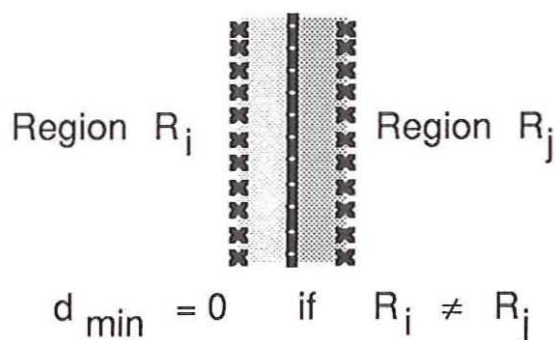


Figure 5.14: Example VI: Modelling of thin conducting foils

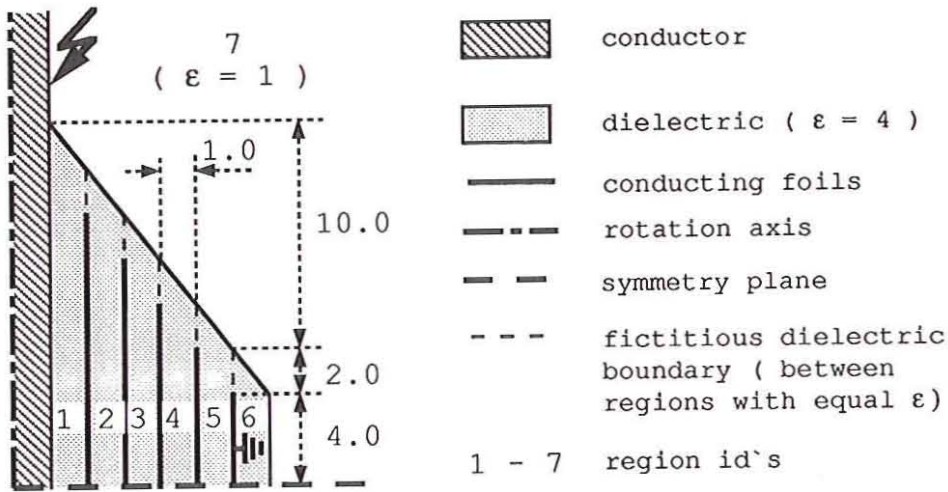


Figure 5.15: Example VI: Model of a HV Bushing

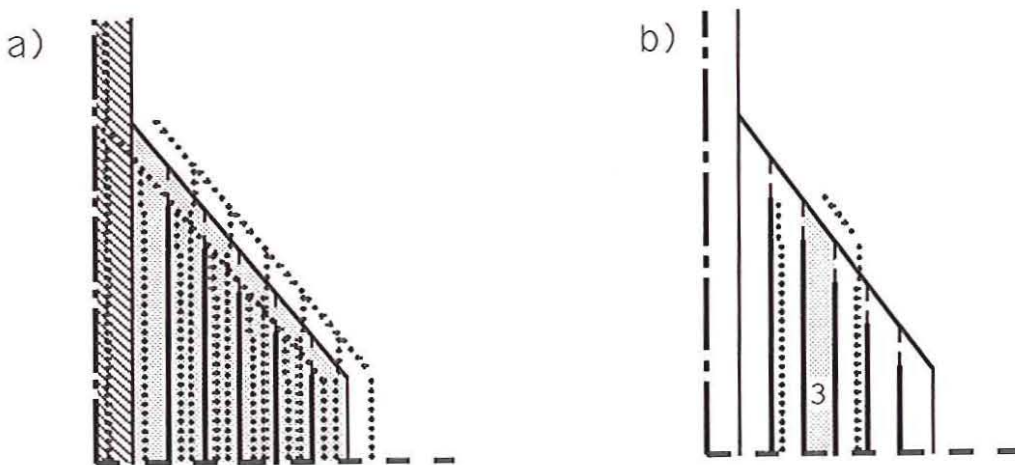


Figure 5.16: Example VI: Charges assigned to:

a) all surfaces

b) surfaces surrounding region 3

be obtained using electrostatic field calculation methods. The results of such calculation for a model of a dry transformer is presented in Fig. 5.18. The voltage wave has been applied on the turn connected directly to the winding entrance. All other turns are for the duration of the wave front on floating potentials. Their values as well as the equipotential lines have been calculated using the charge simulation method (program ELFI). The advantage of this approach is the possibility to obtain the initial insulation stress between turns. In Fig. 5.19 the appropriate vectors of field strength are shown for the part of the winding in the vicinity of the wave entrance point.

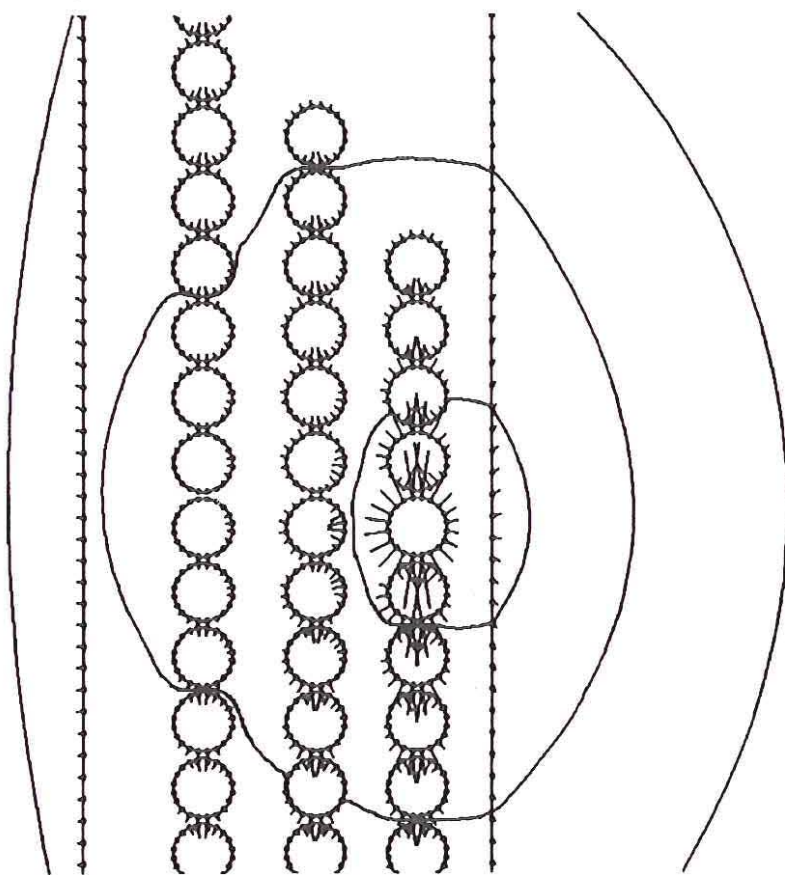


Figure 5.19: Example VII: Field strength vectors in the vicinity of the wave entrance point

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ISBN 3-89336-084-0