COMPUTING ELASTIC INTERIOR TRANSMISSION EIGENVALUES

with the boundary element collocation method

STAIMSR2021 | July 6, 2021 | Andreas Kleefeld (joint work with Maria Zimmermann) | Jülich Supercomputing Centre

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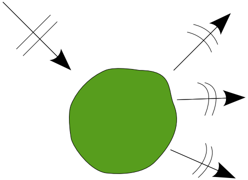
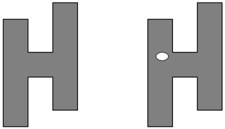
INTRODUCTION

Motivation





Is there an incident field that does not scatter?

Interior transmission eigenvalues (ITEs) $\omega_1, \omega_2, \omega_3, \ldots$ for a homogeneous component are different from a component with an inhomogeneity.
Non-destructive testing

INTRODUCTION

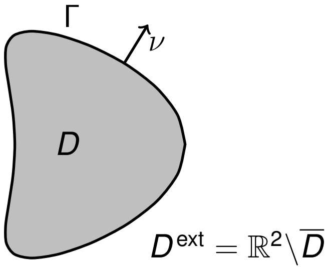
Motivation

- What are elastic interior transmission eigenvalues?
- Can we calculate them numerically?

INTRODUCTION

Problem setup

- D bounded open region in $\mathbb{R}^2$.
- Boundary Γ consists of a finite number of disjoint, closed, bounded surfaces belonging to class C^2 .
- $D^{\text{ext}} = \mathbb{R}^2 \setminus \overline{D}$ is connected.
- ω given frequency.
- ν denotes normal pointing into D^{ext}.
- ϱ_1, ϱ_2 are densities (given constants).
- λ, μ are given Lamé parameters satisfying $\lambda + 2\mu > 0$, $\mu > 0$.
- $\Delta^* u = \mu \, \Delta u + (\lambda + \mu) \, \mathrm{grad} \, \mathrm{div} \, u$



INTRODUCTION

Scattering by an inhomogeneous media

- Solve

$$\begin{aligned} \Delta^* u + \omega^2 \varrho_1 u &= 0 && \text{in } D^{\text{ext}}\,, \\ \Delta^* v + \omega^2 \varrho_2 v &= 0 && \text{in } D\,, \\ u &= v && \text{on } \Gamma\,, \\ T(u) &= T(v) && \text{on } \Gamma\,, \end{aligned}$$

$$\lim_{r\to\infty} \sqrt{r}\,(\partial_r u_p - \mathrm{i}k_p u_p)\,, \quad \lim_{r\to\infty} \sqrt{r}\,(\partial_r u_s - \mathrm{i}k_s u_s) = 0\,, \qquad r = |x|\,.$$

- Total field is $u = u_s + u_p + u_i$ with incident field u_i.
- $T(z) = \lambda\,\mathrm{div}\,(z)\nu + 2\mu\left(\nu^\top \mathrm{grad}\right) z + \mu\,\mathrm{div}(Qz)Q\nu$ with $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- $k_p^2 = \omega^2/(\lambda + 2\mu)$, $k_s^2 = \omega^2/\mu$.
- Is there an incident field that does not scatter?

INTRODUCTION

Elastic interior transmission eigenvalue problem

- Question is related to the elastic interior transmission problem (ITP).
- If u_i is given such that $u_s + u_p|_{\mathbb{R}^2\setminus\overline{D}} = 0$, then setting $w = u|_D$ and $v = u_i|_D$ yields the following problem:
- Find a solution $(v, w) \neq (0, 0)$ to the ITP given by

$$\begin{aligned} \Delta^* w + \omega^2 \varrho_1 w &= 0 && \text{in } D\,, \\ \Delta^* v + \omega^2 \varrho_2 v &= 0 && \text{in } D\,, \\ v &= w && \text{on } \Gamma\,, \\ T(v) &= T(w) && \text{on } \Gamma\,. \end{aligned}$$

- Then $\omega \in \mathbb{C}$ will be an elastic interior transmission eigenvalue (ITE).

INTRODUCTION

History (partial list)

- **Introduction of ITP:**
  Kirsch (1986) and Colton & Monk (1988).
- **Discreteness of ITEs:**
  Colton & Kirsch & Päivärinta (1989), Rynne & Sleeman (1991), Cakoni & Haddar (2007), Colton & Päivärinta & Sylvester (2007), Kirsch (2009), Cakoni & Haddar (2009), and Hickmann (2012).
- **Existence of ITEs:**
  Päivärinta & Sylvester (2009), Kirsch (2009), Cakoni & Gintides & Haddar (2011), Cakoni & Haddar (2011), Cakoni & Kirsch (2011), Bellis & Cakoni & Guzina (2011), and Cossonnière (2011).

INTRODUCTION

Numerical computation of elastic ITEs (recent work)

- **Inside-outside-duality method:** Peters (2016)
- **Method of fundamental solutions (MFS):** Kleefeld & Pieronek (2020)
- **Finite element method (FEM):** Ji & Li & Sun (2018), Xi & Ji (2018), Xi & Ji & Geng (2018), Ji & Li & Sun (2020), Chang & Lin & Wang (2020), Yang & Han & Bi (2020), Yang & Han & Bi & Li & Zhang (2020), and Xi & Ji & Zhang (2021)
- **Boundary element method (BEM):** Weger (2018) and Zimmermann (2021)

SOLVING THE ITP

Boundary integral operators

$$\mathrm{SL}_\kappa(\varphi)(P) = \int_\Gamma \Phi_\kappa(P,q)\varphi(q)\,\mathrm{d}s(q)\,, \qquad P \in D\,,$$

$$\mathrm{DL}_\kappa(\varphi)(P) = \int_\Gamma \left[T_q\left(\Phi_\kappa(P,q)\right)\right]^\top \varphi(q)\,\mathrm{d}s(q)\,, \qquad P \in D\,,$$

$$\mathrm{S}_\kappa(\varphi)(p) = \int_\Gamma \Phi_\kappa(p,q)\varphi(q)\,\mathrm{d}s(q)\,, \qquad p \in \Gamma\,,$$

$$\mathrm{D}_\kappa(\varphi)(p) = \int_\Gamma \left[T_q\left(\Phi_\kappa(p,q)\right)\right]^\top \varphi(q)\,\mathrm{d}s(q)\,, \qquad p \in \Gamma\,,$$

$$\mathrm{D}_\kappa^\top(\varphi)(p) = \int_\Gamma T_p\left(\Phi_\kappa(p,q)\right)\varphi(q)\,\mathrm{d}s(q)\,, \qquad p \in \Gamma\,,$$

and $\Phi_\kappa(p,q)$, $p \neq q$ the fundamental solution.

SOLVING THE ITP

Boundary integral equation

- Assume κ^2 is not a Dirichlet eigenvalue of $-\Delta^*$ in D.
- Dirichlet-to-Neumann operator:

$$N_\kappa = \left(\frac{1}{2}\mathsf{I} + \mathsf{D}_\kappa^\top\right)\mathsf{S}_\kappa^{-1}\,.$$

- Then $M(\omega)v = 0$ solves ITP (see Cakoni & Kress) with

$$M(\omega) = N_{\omega\sqrt{\varrho_1}} - N_{\omega\sqrt{\varrho_2}} = \left(\frac{1}{2}\mathsf{I} + \mathsf{D}_{\omega\sqrt{\varrho_1}}^\top\right)\mathsf{S}_{\omega\sqrt{\varrho_1}}^{-1} - \left(\frac{1}{2}\mathsf{I} + \mathsf{D}_{\omega\sqrt{\varrho_2}}^\top\right)\mathsf{S}_{\omega\sqrt{\varrho_2}}^{-1}\,.$$

- We use

$$M(\omega) = \mathsf{S}_{\omega\sqrt{\varrho_1}}^{-1}\left(\frac{1}{2}\mathsf{I} + \mathsf{D}_{\omega\sqrt{\varrho_1}}\right) - \mathsf{S}_{\omega\sqrt{\varrho_2}}^{-1}\left(\frac{1}{2}\mathsf{I} + \mathsf{D}_{\omega\sqrt{\varrho_2}}\right)\,.$$

SOLVING THE ITP

Boundary integral equation

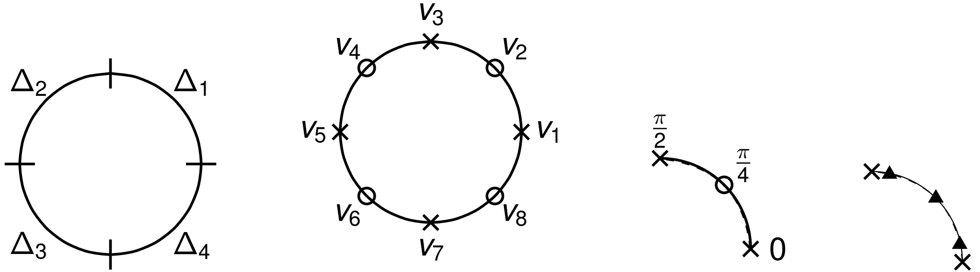
- E is the set of all $\omega^2\varrho_1$ and $\omega^2\varrho_2$ that are Dirichlet eigenvalues of $-\Delta^*$ in D.
- Assume $\omega^2\varrho_1, \omega^2\varrho_2 \notin E$.
- To find ITE, solve the non-linear eigenvalue problem

$$M(\omega)v = 0\,.$$

- M(ω) is Fredholm of index zero.
- M(ω) is analytic on $\mathbb{C}\backslash\{\mathbb{R}^- \cup E\}$.

NUMERICAL SOLUTION

Boundary integral equation



- Subdivide boundary in n_f pieces.
- Define discretization points.
- Approximate boundary pieces.
- Discretize unknown function on each piece.
- Require residual to be zero at $n_c = 3 \cdot n_f$ 'collocation points'.
- Leads to non-linear eigenvalue problem $\mathbf{M}(\omega)\vec{v} = \vec{0}$ with $\mathbf{M}(\omega) \in \mathbb{C}^{2n_c \times 2n_c}$.

NUMERICAL SOLUTION

Solving the non-linear eigenvalue problem

- Consider non-linear eigenvalue problem

$$\mathbf{M}(\omega)\vec{v} = \vec{0}\,, \quad \vec{v} \in \mathbb{C}^{2n_c}\,, \quad \vec{v} \neq 0\,, \quad \omega \in \mathbb{B}(\mu, R) \subset \mathbb{C}\,.$$

- Large scale problem $m \ll 2n_c$ (m is number of eigenvalues including multiplicities).
- Problem can be reduced to linear eigenvalue problem of dimension m (Keldysh's theorem).
- One has to use complex-valued contour integrals.
- See article by W.-J. Beyn (2012).

NUMERICAL RESULTS

Parameters

- $\varrho_1 = 1$, $\varrho_2 = 4$, $\mu = 1/16$, $\lambda = 1/4$
- $N = 24$, $\ell = 20$, $tol = 10^{-2}$, $R = 1/4$, $n_f = 16, 20, 32, 40$.
- D: disk with radius $1/2$, ellipse with semi-axis 1 and 0.5, deformed ellipse (kite), unit square.

NUMERICAL RESULTS

Disk with radius $1/2$

ITE	BEM	FEM [13]	FEM [9]	FEM [4]	MFS [6]
ω_1	1.451 303	1.452 482	1.451 948	1.455 078	1.451 304 028
ω_2	1.704 673	1.706 023	1.705 370	1.709 214	1.704 638 247
ω_3	1.704 674	1.706 023		1.709 214	
ω_4	1.984 555	1.986 143		1.989 630	1.984 530 256
ω_5	1.984 557	1.986 146		1.989 630	
ω_6	2.269 152	2.270 963		2.274 992	2.269 112 085
ω_7	2.269 156				

- BEM yields comparable results to MFS.
- Using only $n_f = 20$ (for ω_1, ω_2, and ω_3) and $n_f = 40$ (for ω_4, ω_5, ω_6, and ω_7) yields better results than FEM [9] ($h = 1/160$), FEM [4] ($h = 1/80$), and FEM [13] ($h \approx 0.03125$). FEM [10] ($h = 0.0125$) yields 1.456.
- Remark: FEM [9] converges numerically with order one, but they state order two.

NUMERICAL RESULTS

Ellipse with radius semi-axis 1 and 1/2

ITE	BEM	MFS [6]
ω_1	1.296 779	1.296 728 137
ω_2	1.302 946	1.302 785 814
ω_3	1.540 739	1.540 896 035
ω_4	1.565 357	1.565 151 107

- Comparable results to MFS.
- Used only $n_f = 20$.

NUMERICAL RESULTS

Kite (deformed ellipse)

ITE	BEM	MFS [6]
ω_1	0.947 094	0.947
ω_2	1.047 417	1.047
ω_3	1.111 296	1.111
ω_4	1.235 417	1.235

- Better results than MFS.
- Used only $n_f = 20$.
- BEM better for general domains D.

NUMERICAL RESULTS

Unit square

ITE	BEM	FEM [13]	FEM [4]	FEM [10]	FEM [9]	MFS [6]
ω_1	1.393 770	1.393 877	1.393 874	1.393 879	1.394 419	1.393 8
ω_2	1.618 379	1.618 299	1.618 296		1.619 008	1.618 2
ω_3	1.618 379	1.618 299	1.618 296			
ω_4	1.801 996	1.802 042	1.802 032			1.802 0
ω_5	1.936 157	1.936 138	1.936 134			1.936 2

- BEM yields better results than FEM [9] ($h = 0.00625$).
- Used only $n_f = 16$ and $n_f = 32$ for ω_5.
- FEM [13] ($h \approx 0.03125$), FEM [4] ($h \approx 0.025$), FEM [10] ($h = 0.0125$), and FEM [12] ($m = 26$) better than BEM.

NUMERICAL RESULTS

Complex ITEs

D	BEM
Circle	$1.987\,189 + 0.283\,145\mathrm{i}$
Unit square	$1.866\,002 + 0.291\,556\mathrm{i}$

- Used only $n_f = 20$ and $n_f = 16$, respectively.

SUMMARY AND OUTLOOK

- Presented an alternative method to calculate ITEs for various domains in 2D.
- Used boundary integral equations.
- Results are very accurate with less computational cost.
- Complex-valued ITEs can be calculated.

- Further investigation is needed for the complex-valued ITEs.
- Likewise exterior transmission eigenvalues can be computed.

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CONTACT INFORMATION

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