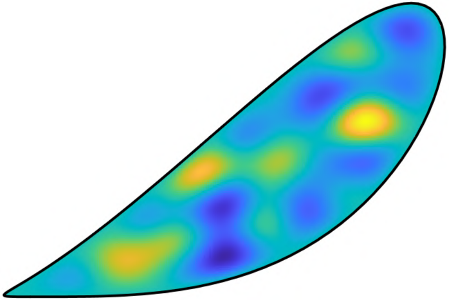
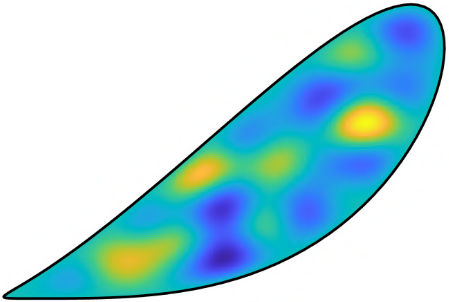


NUMERICAL SOLUTION OF TWO-DIMENSIONAL SPDES

using a Galerkin exponential time differencing scheme

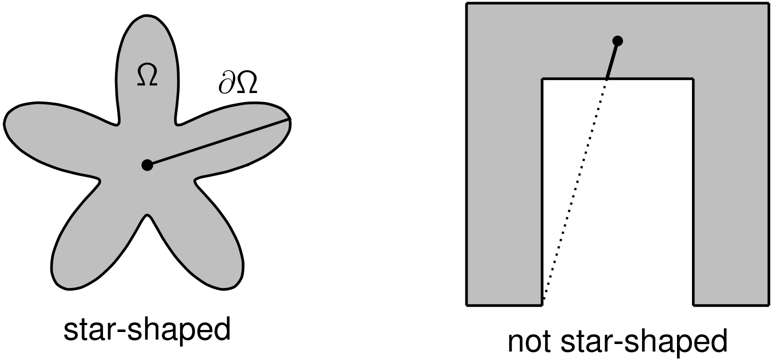
NUMDIFF-16 | September 9, 2021 | Julian Clausnitzer | Jülich Supercomputing Centre, Germany

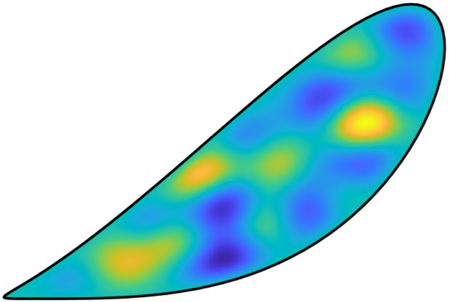


Part I: Introduction & motivation

INTRODUCTION AND OVERVIEW

- Aim of the project (joint work with Andreas Kleefeld): Solving parabolic stochastic partial differential equations (SPDEs) on star-shaped Lipschitz two-dimensional domains.
- What's new about this: Until now, only relatively simple domains have been considered.



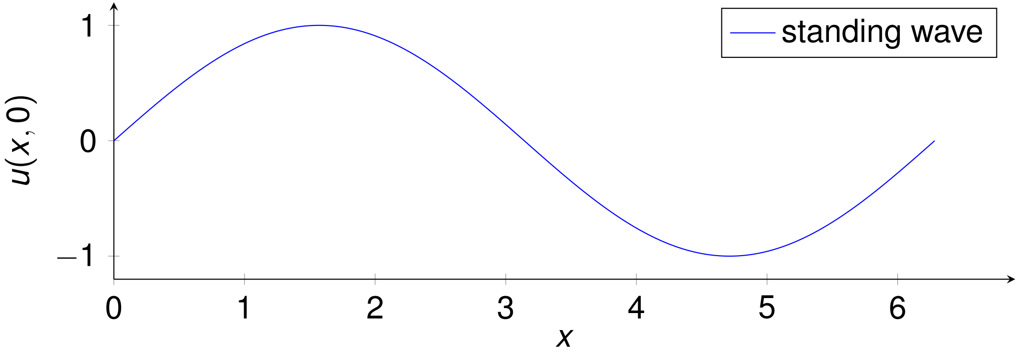


Part II: Theory

THEORY
SPDEs

- Solution of a partial differential equation (PDE): A function. E.g. the vibrating string of a guitar for the wave equation.

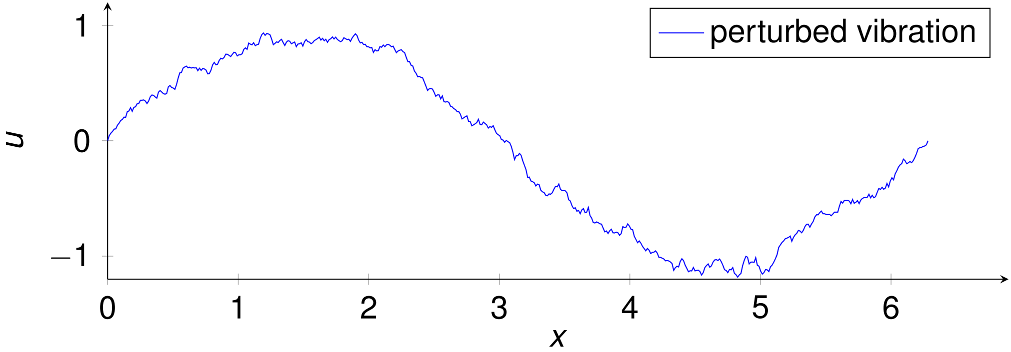
$$\partial_{tt} u(x,t) = c^2 \partial_{xx} u(x,t)$$



THEORY
SPDEs

- Solution of an SPDE: A stochastic process. E.g. a vibrating string fluctuating randomly.

$$\partial_{tt} u(x,t) = c^2 \partial_{xx} u(x,t) + \xi_t$$



THEORY
Setting

1. Stochastic analysis and Wiener processes in Hilbert spaces
2. Approximation of SPDEs: Projection and temporal discretization
3. Boundary integral formulation and Beyn integral algorithm for more general 2D domains

THEORY

1. Stochastic analysis

In SPDEs, the process X_t does not take values in $\mathbb{R}$, but in a general Hilbert space H (here, e.g. $H = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$). We consider

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + \mathrm{d}W(t), \quad X(0) = x_0$$

or, equivalently:

$$X(t) = x_0 + \int_0^t AX(s)\mathrm{d}s + \int_0^t \mathrm{d}W^Q(s),$$

where

1. $(X(t))_{t\in[0,T]}$, $T > 0$, is a stochastic process taking values in H,
2. $A : H \to H$ is a linear operator (typically, $A = \Delta$), and generates an analytical semigroup $(S(t))_{t\in[0,T]}$.
3. $(W^Q(t))_{t\in[0,T]}$ a Wiener process (Brownian motion) taking values in H.

The $\mathrm{d}W^Q(s)$ integral is called stochastic integral.

Under the

THEORY

1. Stochastic analysis: Wiener processes

The H-valued Q-cylindrical Wiener process $(W(t))_{t\in[0,T]}$ has a series representation

$$W^Q(t) = \sum_{i=1}^{\infty} \sqrt{q_i}\beta_i(t)\phi_i, \tag{1}$$

with $q_i \geq 0$ are the eigenvalues of Q, $\{\phi_i\}_{i\in\mathbb{N}}$ an ONB of H (usually assumed to be the eigenfunctions of A) and β_k one-dimensional Brownian motions.
To satisfy the condition (Jentzen, Kloeden 2008)

$$\sum_{i=1}^{\infty}(\lambda_i)^{2\gamma-1}q_i < \infty, \tag{2}$$

in the 2D case (where $\lambda_N \sim N$ are Dirichlet eigenvalues and $\gamma \in (0,1)$) it must be assumed that $q_i \sim i^{-\eta}$ for $\eta > 0$ or that $q_i = 0$ for some $i > K \in \mathbb{N}$ (smooth noise).

The mild solution of the SPDE is given by the implicit equation

$$X(t) = S(t)x_0 + \int_0^t S(t-s)\,\mathrm{d}W^Q(s) \tag{3}$$

with $S(t) = \mathrm{e}^{At}$ the analytic semigroup.
Under these assumptions (and possibly, suitable assumptions on a nonlinearity F), there exists a unique mild solution (Jentzen, Kloeden 2008).

THEORY

2. Approximation

The approximation of the solution is done in two steps:

1. Projection onto a finite-dimensional subspace of H.
2. Discretization in time.

THEORY

2. Approximation: Projection

For our equations, we assume Dirichlet boundary conditions, i.e.,

$$X|_{\partial\Omega} = 0.$$

Galerkin approach: The solution $(X(t))_{t\in[0,T]}$ is extended in a Fourier-like series, the base functions being the Dirichlet eigenfunctions u of Δ in Ω satisfying

$$\Delta u = \lambda u \tag{4}$$

for a real number $\lambda \in \mathbb{R}_{>0}$. They form an orthonormal basis (ONB) of $L^2(\Omega)$.

THEORY

2. Approximation: Spatial Projection

There is an orthonormal basis $\{\phi_i\}_{i\in\mathbb{N}}$ of $L^2(\Omega)$ satisfying

$$\Delta\phi_i = \lambda_i\phi_i, \quad i \in \mathbb{N}$$

for real eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. We use a projection

$$P_N : H \to H_N = \mathrm{span}\{\phi_1, \ldots, \phi_N\}$$

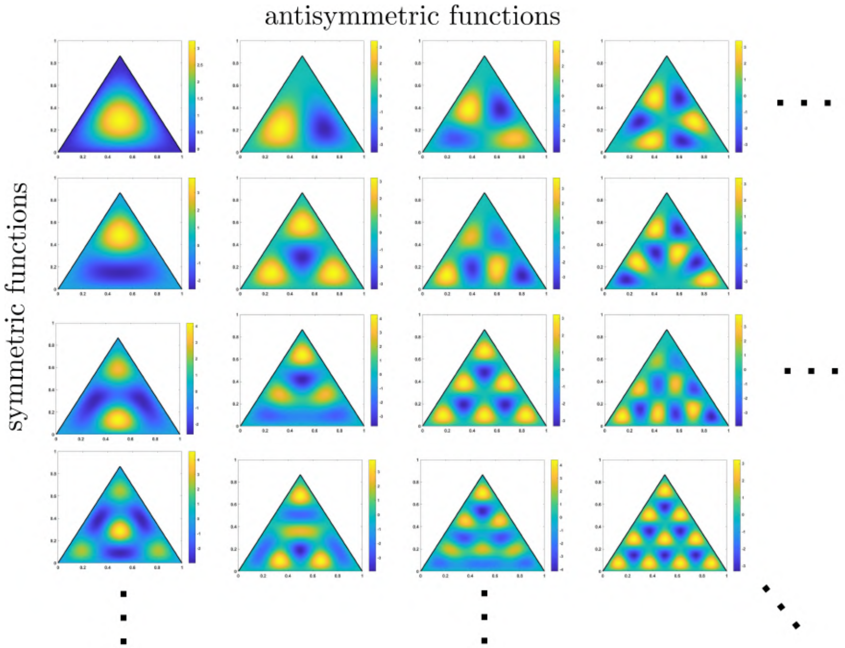
$$u(x) = \sum_{i=1}^{\infty} \langle \phi_i, u\rangle \phi_i(x) \mapsto \sum_{i=1}^{N} \langle \phi_i, u\rangle \phi_i(x)$$

and consider the projected equation

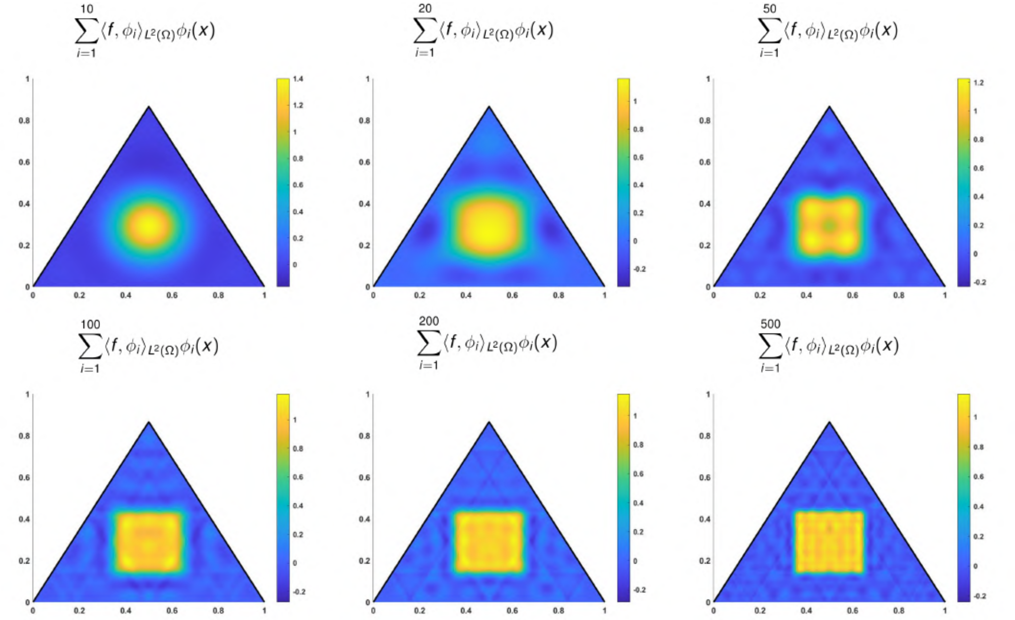
$$\mathrm{d}X_N(t) = A_N X_N(t)\mathrm{d}t + \mathrm{d}W_N(t), \quad X_N(0) = x_0,$$

$X_N = P_N \circ X,\ A_N = P_N \circ A,\ W_N = P_N \circ W.$

Orthonormal basis for $L^2(\Omega)$, where Ω is an equilateral triangle:



Fourier approximations for a centered square in Ω, $f(x) = \mathbb{1}_{[0.35,0.65]\times[0.139,0.439]}(x)$



THEORY

2. Approximation: Temporal discretization

$$\mathrm{d}X_N(t) = A_N X_N(t)\mathrm{d}t + \mathrm{d}W_N(t), \quad X_N(0) = x_0 \tag{5}$$

We consider a running Fourier expansion

$$X_N(t) = \sum_{i=1}^{N} v_i^N(t) \cdot \phi_i.$$

We divide (5) into M time steps, step size $h = T/M$, $t_j = j \cdot h$, $j = 1, \dots, M$.

We consider a running Fourier expansion

$$X_N(t) = \sum_{i=1}^{N} v_i^N(t) \cdot \phi_i.$$

The full scheme for the coefficients $v_{i,j}^{N,M} = v_i^N(t_j)$, $t_j = j \cdot \frac{T}{M}$, is (Jentzen, Kloeden 2008)

$$v_{i,j+1}^{N,M} = \exp(-\lambda_i h) v_{i,j}^{N,M} + \left(\frac{1}{2\lambda_i}(1 - \exp(-2\lambda_i h))\right)^{1/2} R_{i,j},$$

where $R_{i,j} \sim N(0, 1)$ i.i.d., $i = 1, \dots, N$, $j = 0, \dots, M - 1$.

THEORY

3. Boundary integral formulation and Beyn integral algorithm

For the presented scheme, the eigenvalues λ_i and eigenfunctions ϕ_i, $i = 1, \dots, N$, are needed.

- For simple shapes (disk, square, triangle), λ_i and ϕ_i are known exactly
- For more general domains: The eigenvalue problem

$$\begin{aligned} \Delta u + \kappa^2 u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &\neq 0. \end{aligned}$$

  is solved with a double-layer ansatz

$$u(x) = \mathrm{DL}_\kappa[\psi](x) = \int_{\partial\Omega} \psi(y) \cdot \partial_{\nu(y)} \Phi_\kappa(x, y)\, \mathrm{d}s(y).$$

THEORY

3. Boundary integral formulation and Beyn integral algorithm

We use a double layer ansatz

$$u(x) = \mathrm{DL}_\kappa[\psi](x) = \int_{\partial\Omega} \psi(y) \cdot \partial_{\nu(y)} \Phi_\kappa(x, y) \,\mathrm{d}s(y), \quad x \in \Omega$$

with a density function $\psi \in C(\partial\Omega)$ and the fundamental solution

$$\Phi_\kappa(x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa \|x - y\|). \tag{6}$$

Using the jump relation of the double layer potential, we have the equation

$$-\frac{1}{2}\psi(x) + \int_{\partial\Omega} \psi(y) \cdot \partial_{\nu(y)} \Phi_\kappa(x, y) \,\mathrm{d}s(y) = 0, \quad x \in \partial\Omega \tag{7}$$

which is discretized using a boundary element collocation method. This gives rise to a nonlinear eigenvalue problem

$$M(\kappa)\vec{\psi} = 0, \tag{8}$$

where $\vec{\psi}$ contains the values of ψ at the collocation nodes.

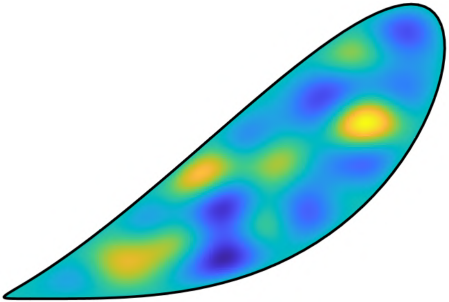
The nonlinear eigenvalue problem

$$M(\kappa)\vec{\psi} = 0 \tag{9}$$

is solved using the Beyn algorithm (Beyn, 2012), which makes use of Keldysh's theorem from complex analysis to retrieve the eigenvalues from contour integrals of the form

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z) M(z)^{-1} \mathrm{d}z, \tag{10}$$

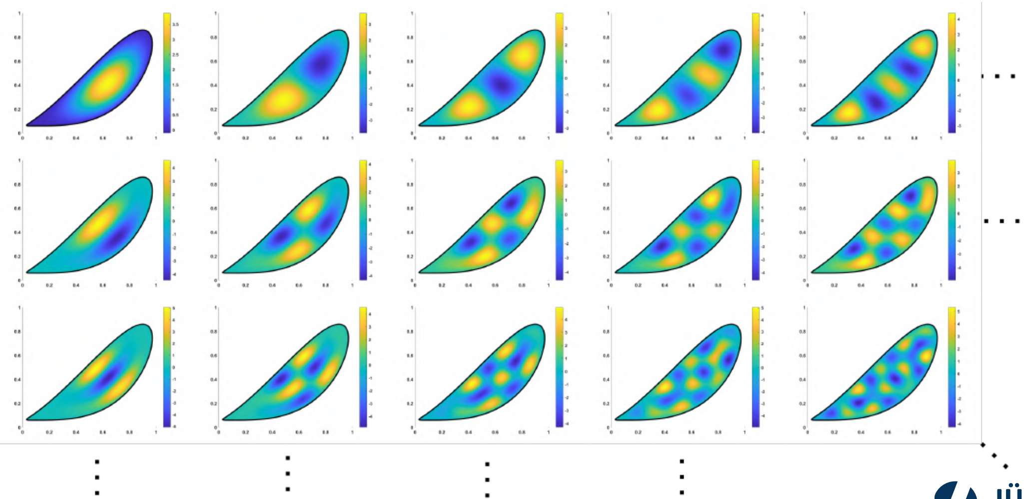
where f is a holomorphic function and γ is a contour containing an eigenvalue.



Part III: Numerical results

We show the first elements of the ONB for $L^2(\Omega)$ for the wing-shaped domain whose boundary inside $[0,1] \times [0,1]$ is given by

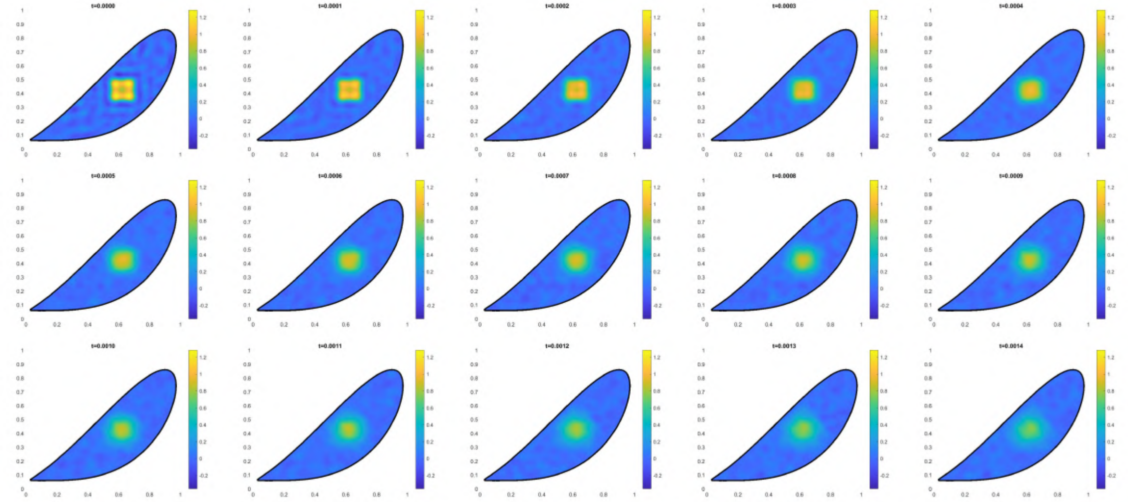
$$\gamma(t) = \begin{pmatrix} 0.25\sin(t) - 0.4\cos(t) + 0.5 \\ 0.2(\cos(t) - 1)^2 + 0.3 \end{pmatrix}, \quad t \in [0, 2\pi). \tag{11}$$



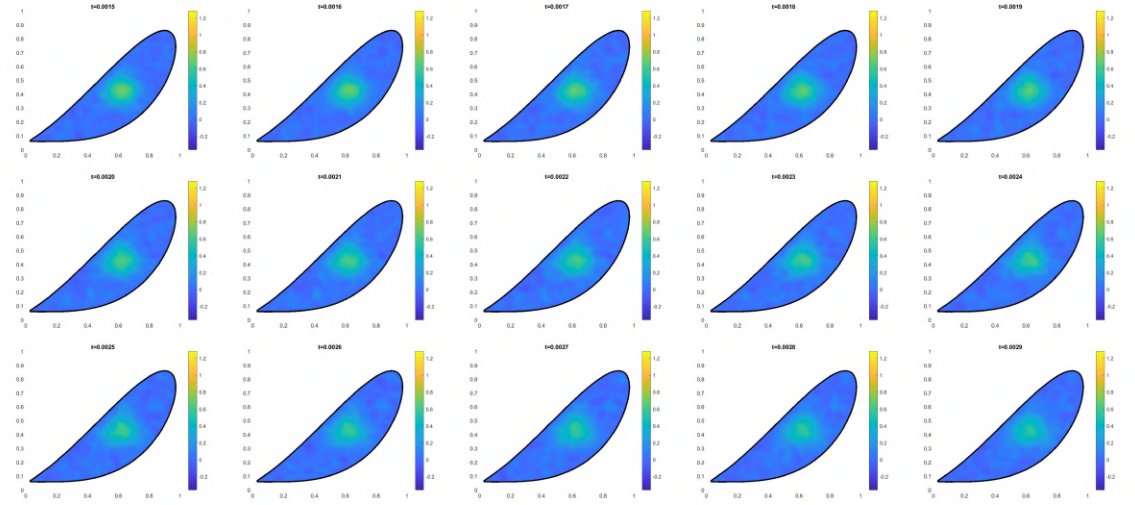
A plot for the heat equation

$$\mathrm{d}X(t) = \Delta X(t)\mathrm{d}t + 0.1\mathrm{d}W(t), \quad X(0) = \mathbb{1}_{[0.55,0.7]\times[0.35,0.5]}, \; t \in [0, 0.01] \qquad (12)$$

with $N = 200$ eigenfunctions and $M = 100$ time steps:



For $t \in [0.0015, 0.0029]$



NUMERICAL RESULTS

Error analysis

We perform the Galerkin-ETD scheme with perturbed eigenvalues and eigenfunctions. Hence, there are three error sources:

1. The error of the Galerkin-ETD scheme,
2. The approximation of the eigenvalues in the Galerkin scheme,
3. The approximation of the eigenfunctions in the Galerkin scheme.

NUMERICAL RESULTS

Error analysis

The following can be said about the errors:

1. The error of the Galerkin-ETD scheme: This error has been shown to be (Jentzen, Kloeden 2008)

$$\sup_{j=0,\ldots,M} \left(\mathbb{E} \| X(t_j) - \widetilde{X}_{M,N}(t_j) \|^2_{L^2(\Omega)} \right)^{1/2} \leq C_T \left(\lambda_N^{\epsilon-1} + \frac{\log(M)}{M} \right) \qquad (13)$$

   for any $\epsilon > 0$ (spatial error, temporal error). This means that the scheme should have a strong convergence order of $1 - \epsilon$ for any $\epsilon > 0$ w.r.t. λ_N and M. (In two dimensions, $\lambda_N \sim N$ asymptotically.)

NUMERICAL RESULTS

Error analysis

2. Approximate eigenvalue error: For an error tolerance

$$\tilde{\epsilon} = \max_{i=1,\dots,N} \{|\lambda_i - \tilde{\lambda}_i|\} \tag{14}$$

   between the approximate eigenvalues $\tilde{\lambda}_i$ and the exact eigenvalues λ_i, it can be shown that the resulting error behaves as $\mathcal{O}(\tilde{\epsilon} \cdot M \cdot N)$.

3. Approximate eigenfunction error: Likewise, for an error tolerance

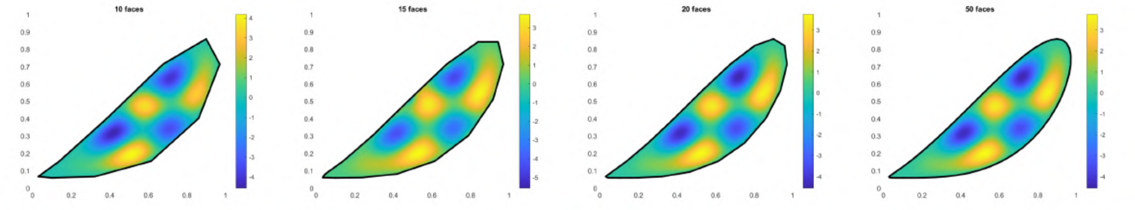
$$\tilde{\epsilon} = \max_{i=1,\dots,N} \{\|e_i - \tilde{e}_i\|_{L^2(\Omega)}\} \tag{15}$$

   between the approximate eigenfunctions $\tilde{e}_i$ and the exact eigenfunctions e_i, it can be shown that the resulting error also behaves as $\mathcal{O}(\tilde{\epsilon} \cdot M \cdot N)$.

NUMERICAL RESULTS

Error analysis

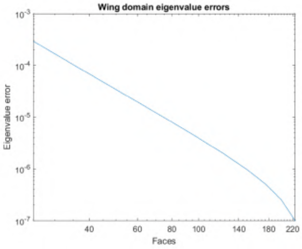
Note that the first error term 'competes' against the other two in the sense that it decreases as M, N grow, while the other two grow as M, N grow.
This necessitates a high accuracy for the eigenvalues and eigenfunctions. The accuracy depends on the number of boundary elements.

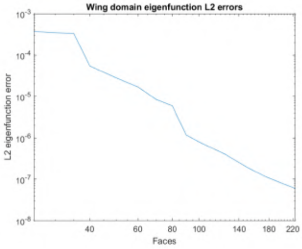


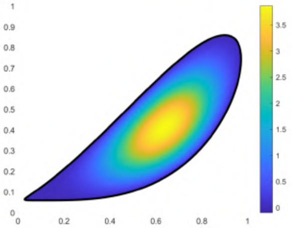
From high-frequency scattering theory (e.g. Bruno, Reitich 2008) it is known that to maintain a given accuracy, the number n_f of boundary elements must rise linearly with $\kappa_N = \sqrt{\lambda_N}$.

Absolute eigenvalue errors and L^2 eigenfunction errors to a reference for $n_f = 250$ faces for the wing domain:

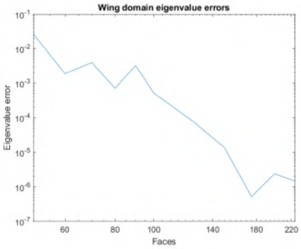
Wave number $\kappa_1 = \sqrt{\lambda_1} \approx 9.7614$:

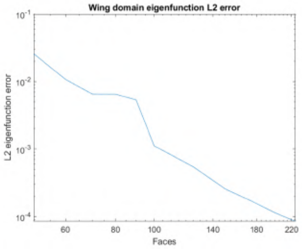


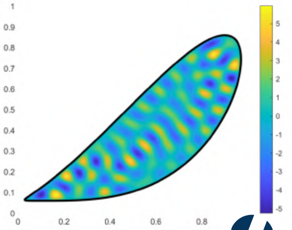




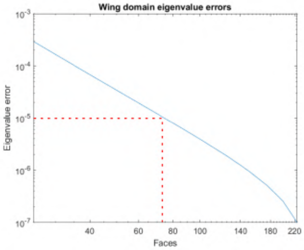
Wave number $\kappa_{112} = \sqrt{\lambda_{112}} \approx 70.9452$:

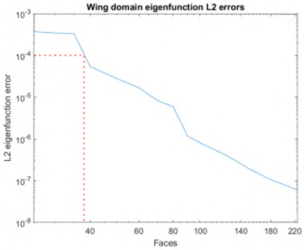


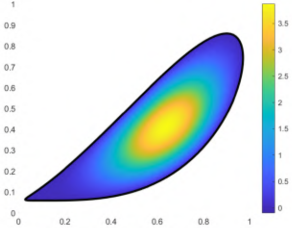




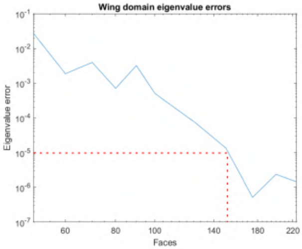
Suppose that we set as tolerances $\epsilon_1 = 10^{-5}$ for the eigenvalues and $\epsilon_2 = 10^{-4}$ for the eigenfunctions. Wave number $\kappa_1 = \sqrt{\lambda_1} \approx 9.7614$:

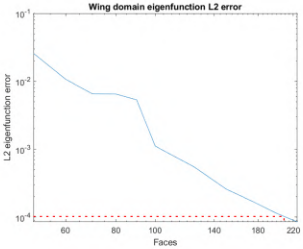


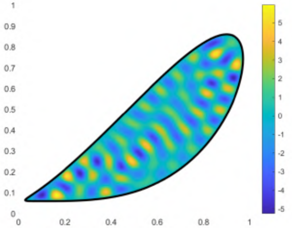




Wave number $\kappa_{112} = \sqrt{\lambda_{112}} \approx 70.9452$:



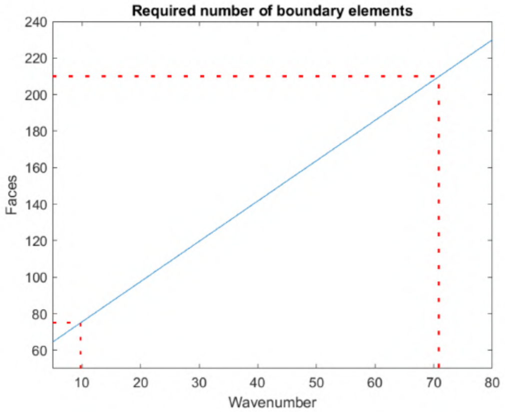




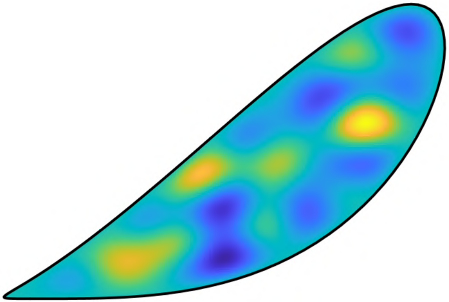
We create a linear fit between the found required face numbers

$$\kappa_1 \approx 9.7614, \quad n_f(\kappa_1) = 75, \tag{16}$$

$$\kappa_2 \approx 70.9452, \; n_f(\kappa_{112}) = 210. \tag{17}$$



In this way, we can compute eigenvalues and an ONB to a prescribed accuracy for any domain.



Part IV: Outlook, references

OUTLOOK, REFERENCES

Other aspects:

- Effect of nonlinear functions in the SPDE (e.g. reaction-diffusion equations) (work in progress)
- Parallelization of the implementation (work in progress)
- Possible applications: Investigation of pattern formation under perturbations
- Adaptive time-stepping schemes

References, further reading:

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- O. P. Bruno and F. Reitich. High-order methods for high-frequency scattering applications. *Modeling and Computations in Electromagnetics.* Springer, 2008. 129–163.

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