Polarization measurements for Electric Dipole Moment and Axion/ALP searches

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PhD School & Workshop Aspects of Symmetries, Nov. 2021

Outline

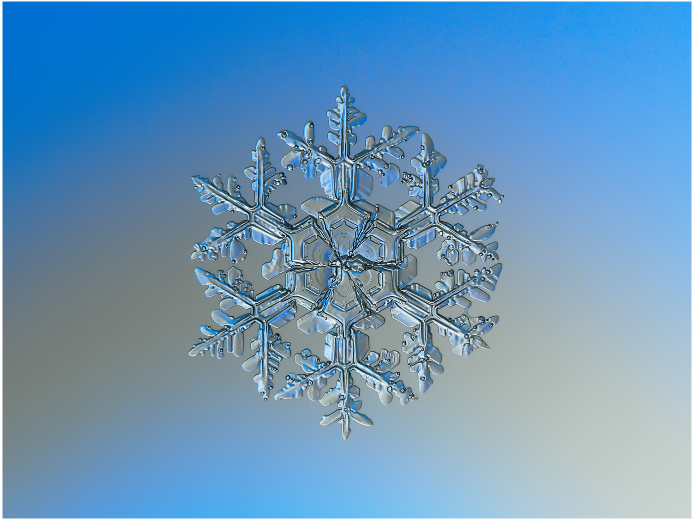
- **Symmetries**
- **Electric Dipole Moments** (EDM)
- $\Rightarrow$ Observable: **Polarisation**
  Optimal Observables, Event Weighting, Maximum Likelihood Method
- **Axion** searches at storage rings
- $\Rightarrow$ How to set **upper limits** if you don't see a signal? Feldman-Cousins algorithm

Symmetries

Symmetries . . .

= invariance under transformations (rotation, translation, reflection)

. . . play an important role in physics



sources: https://commons.wikimedia.org

Symmetries . . .

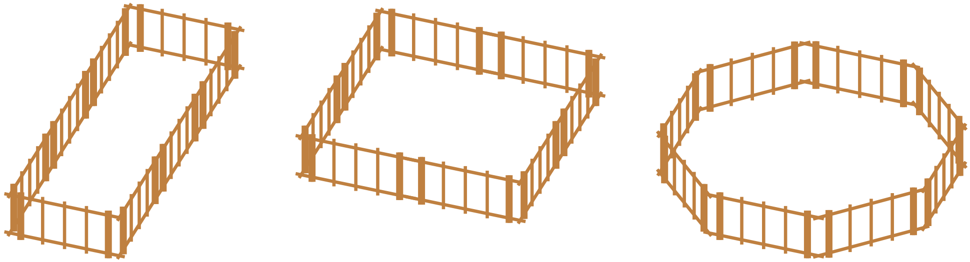
. . . have esthetic aspects



sources: Wikipedia, Stadt Aachen, https://www.fotocommunity.de

Symmetries . . .

have also practical aspects



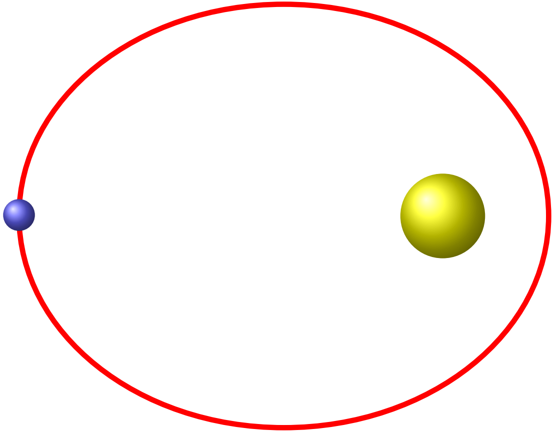
Fundamental symmetries in physics

- **Parity** $\mathcal{P}$ (point reflection)
- **Time reversal** $\mathcal{T}$ (process runs backwards)
- **Charge conjugation** $\mathcal{C}$ (exchange particle and anti-particle)

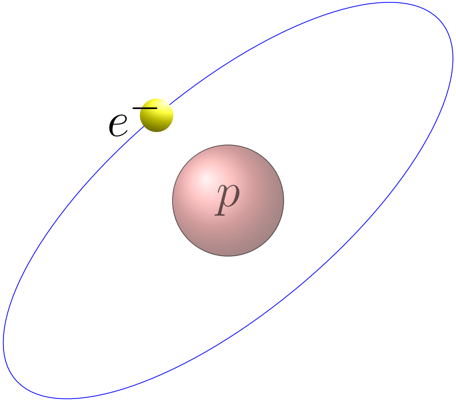
Parity $\mathcal{P}$

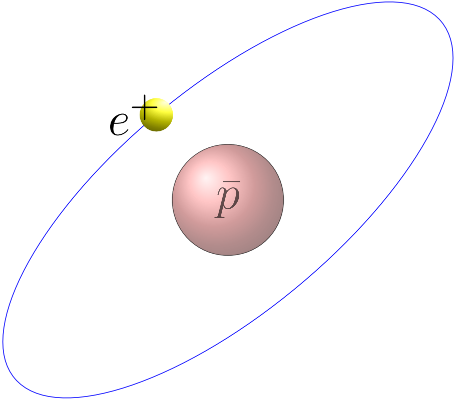
Parity $\mathcal{P}$

Time reversal $\mathcal{T}$



Charge conjugation $\mathcal{C}$: Matter Anti-matter asymmetry





matter:
abundent on earth

Anti-matter:
only produced in accelerators

$\Rightarrow$ Large Asymmetry between matter and anti-matter

matter – anti-matter asymmetry

- According to our present knowledge in the early universe matter and anti-matter were equally present
- today we are surrounded by matter
- Where is the anti-matter?

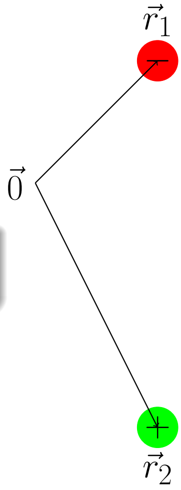
- Which mechanisms caused the disappearance of anti-matter?
- Are there "Anti-worlds"?

Electric Dipole Moments

Electric Dipoles

Classical definition:

$$\vec{d} = \sum_i q_i \vec{r}_i$$



Order of magnitude

	atomic physics	hadron physics
charges	e	
$\lvert \vec{r}_1 - \vec{r}_2 \rvert$	1 Å$= 10^{-8}$cm	
EDM		
naive expectation	$10^{-8} e \cdot \mathrm{cm}$	
observed	water molecule	
	$4 \cdot 10^{-9} e \cdot$ cm	

Order of magnitude

	atomic physics	**hadron physics**
charges	e	e
$\lvert\vec{r}_1 - \vec{r}_2\rvert$	1 Å= 10^{-8}cm	$1\,\mathrm{fm} = 10^{-13}$cm
EDM		
naive expectation	$10^{-8} e \cdot \mathrm{cm}$	$10^{-13} e \cdot \mathrm{cm}$
observed	water molecule	neutron
	$4 \cdot 10^{-9} e \cdot$ cm	$< 3 \cdot 10^{-26} e \cdot$ cm

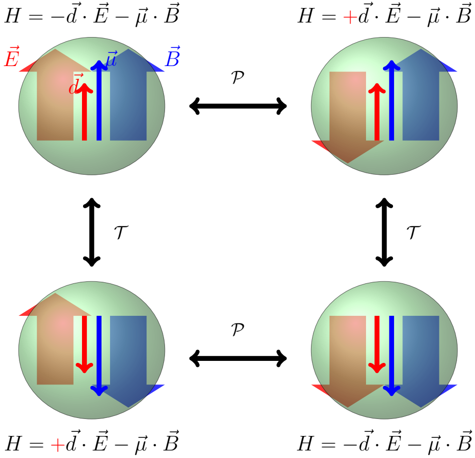
EDM Operator

	E (electric field)	P odd	
classical:	$\vec{d} = e\vec{r}$	P odd	large EDM possible, e.g. molecules with
	$H = -\vec{d} \cdot \vec{E}$	P even	degenerated ground states of different parity
spin	$\vec{d} = d\vec{s}/\|\vec{s}\|$	P even	
	$H = -\vec{d} \cdot \vec{E}$	P odd	EDM possible if P (and T) violated

$\mathcal{T}$ and $\mathcal{P}$ violation of EDM

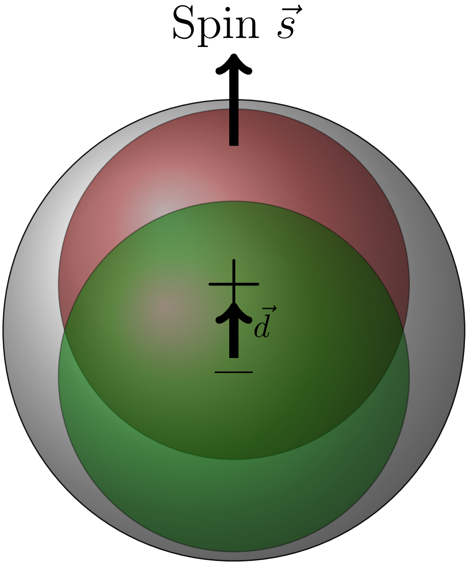
$\vec{d}$: EDM
$\vec{\mu}$: magnetic moment (MDM)
both || to spin $\vec{s}$

	H	$=$	$-\mu\frac{\vec{s}}{s}\cdot\vec{B}-d\frac{\vec{s}}{s}\cdot\vec{E}$
$\mathcal{T}$:	H	$=$	$-\mu\frac{\vec{s}}{s}\cdot\vec{B}+d\frac{\vec{s}}{s}\cdot\vec{E}$
$\mathcal{P}$:	H	$=$	$-\mu\frac{\vec{s}}{s}\cdot\vec{B}+d\frac{\vec{s}}{s}\cdot\vec{E}$



$\Rightarrow$ EDM measurement tests violation of fundamental symmetries $\mathcal{P}$ and $\mathcal{T}(\overset{\mathcal{CPT}}{=}\mathcal{CP})$

Electric Dipole Moments (EDM)

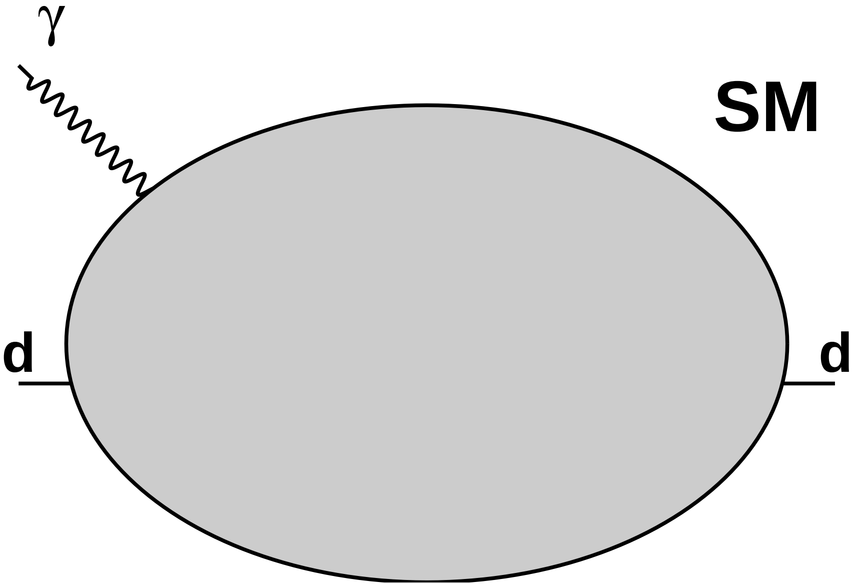


- permanent separation of positive and negative charge
- fundamental property of particles (like magnetic moment, mass, charge)
- existence of EDM only possible via violation of time reversal $\mathcal{T} \overset{\mathcal{CPT}}{=} \mathcal{CP}$ and parity $\mathcal{P}$ symmetry
- close connection to "matter-antimatter" asymmetry
- axion field leads to oscillating EDM

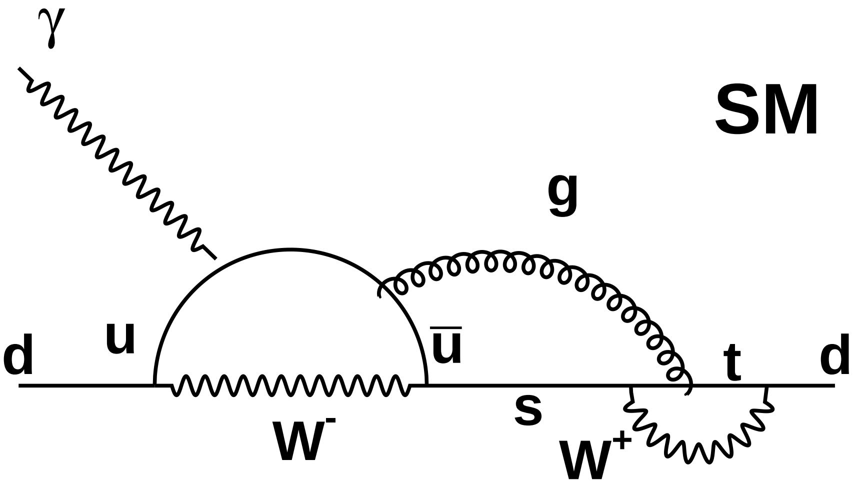
$\mathcal{CP}-$Violation & connection to EDMs

Standard Model	
Weak interaction	
CKM matrix	$\rightarrow$ unobservably small EDMs
Strong interaction	
θ_{QCD}	$\rightarrow$ best limit from neutron EDM
beyond Standard Model	
e.g. SUSY	$\rightarrow$ accessible by EDM measurements

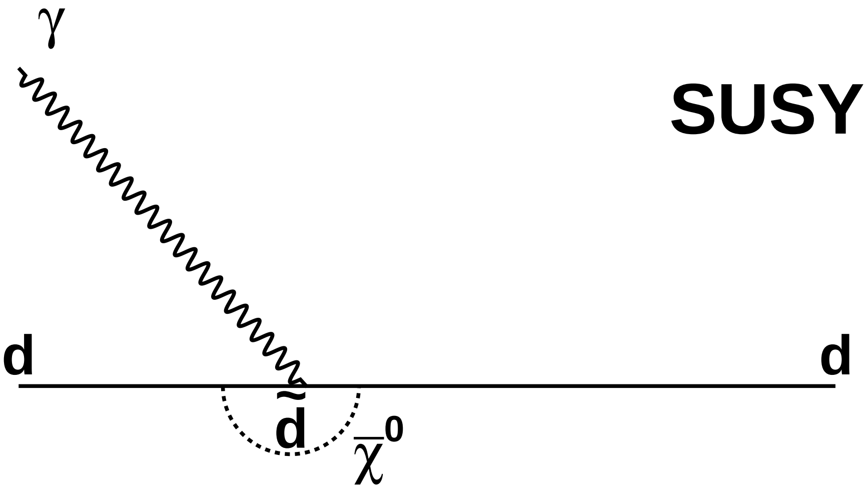
EDM in SM and SUSY



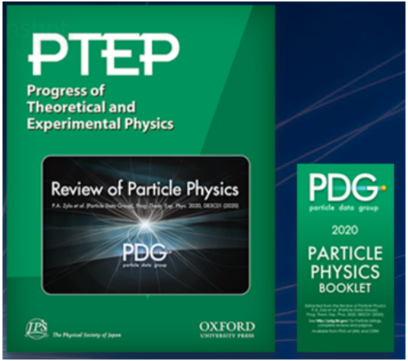
EDM in SM and SUSY



EDM in SM and SUSY



Proton EDM



Citation: P.A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020) and 2021 update

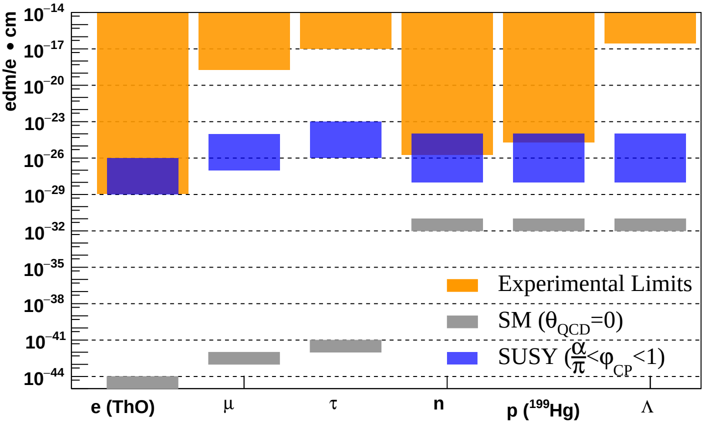
N BARYONS $(S = 0,\ I = 1/2)$

$p,\ N^+ = uud; \quad n,\ N^0 = udd$

p $\qquad I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

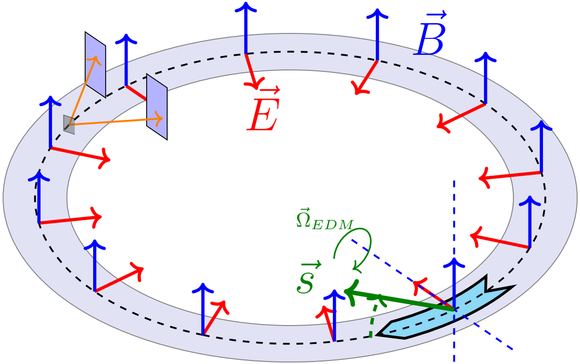
Mass $m = 1.00727646663 \pm 0.00000000009$ u $\quad (S = 2.9)$
Mass $m = 938.272081 \pm 0.000006$ MeV [a]
$|m_p - m_{\overline{p}}|/m_p < 7 \times 10^{-10}$, CL = 90% [b]
$|\frac{q_{\overline{p}}}{m_{\overline{p}}}|/(\frac{q_p}{m_p}) = 1.00000000000 \pm 0.00000000007$
$|q_p + q_{\overline{p}}|/e < 7 \times 10^{-10}$, CL = 90% [b]
$|q_p + q_e|/e < 1 \times 10^{-21}$ [c]
Magnetic moment $\mu = 2.7928473446 \pm 0.0000000008\ \mu_N$
$(\mu_p + \mu_{\overline{p}})\ /\ \mu_p = (0.002 \pm 0.004) \times 10^{-6}$
Electric dipole moment $d < 0.021 \times 10^{-23}\ e\,$cm
Electric polarizability $\alpha = (11.2 \pm 0.4) \times 10^{-4}$ fm^3
Magnetic polarizability $\beta = (2.5 \pm 0.4) \times 10^{-4}$ fm^3 $\quad (S = 1.2)$
Charge radius, μp Lamb shift $= 0.84087 \pm 0.00039$ fm [d]
Charge radius $= 0.8409 \pm 0.0004$ fm [d]
Magnetic radius $= 0.851 \pm 0.026$ fm [e]
Mean life $\tau > 3.6 \times 10^{29}$ years, CL = 90% [f] $\quad (p \to$ invisible mode)
Mean life $\tau > 10^{31}$ to 10^{33} years [f] $\quad$ (mode dependent)

EDM: Current Upper Limits



storage rings: EDMs of **charged** hadrons: p, d, ^{3}He

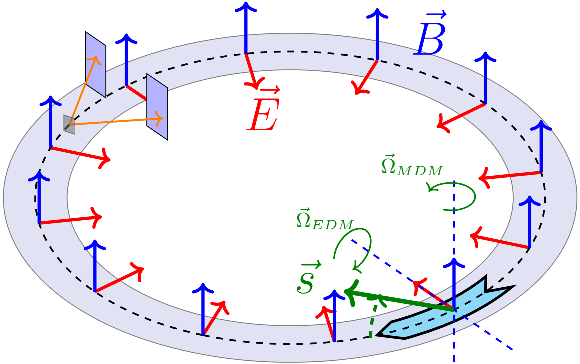
Experimental Method: Generic Idea



$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} \propto \underbrace{d(\vec{E} + \vec{v} \times \vec{B})}_{= \vec{\Omega}_{\mathrm{EDM}}} \times \vec{s}$$

build-up of vertical polarization $s_\perp \propto d$, if $\vec{s}_{\mathrm{horz}} || \vec{p}$ (**frozen spin**)

Experimental Method: Generic Idea



$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} \propto \underbrace{d(\vec{E} + \vec{v} \times \vec{B})}_{=\vec{\Omega}_{\text{EDM}}} \times \vec{s}$$

In general:

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = (\vec{\Omega}_{\text{MDM}} + \vec{\Omega}_{\text{EDM}}) \times \vec{s}$$

build-up of vertical polarization $s_\perp \propto d$, if $\vec{s}_{\text{horz}} || \vec{p}$ (**frozen spin**)

Spin Precession: Thomas-BMT Equation

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{\Omega} \times \vec{s} = \frac{-q}{m}\left[\underbrace{G\vec{B} + \left(G - \frac{1}{\gamma^2 - 1}\right)\vec{v} \times \vec{E}}_{=\vec{\Omega}_{\mathrm{MDM}}} + \underbrace{\frac{\eta}{2}(\vec{E} + \vec{v} \times \vec{B})}_{=\vec{\Omega}_{\mathrm{EDM}}}\right] \times \vec{s}$$

electric dipole moment (EDM): $\vec{d} = \eta \frac{q\hbar}{2mc}\vec{s}$,

magnetic dipole moment (MDM): $\vec{\mu} = 2(G+1)\frac{q\hbar}{2m}\vec{s}$

Note: $\eta = 2 \cdot 10^{-15}$ for $d = 10^{-29}$ ecm, $G \approx 1.79$ for protons

Spin Precession: Thomas-BMT Equation

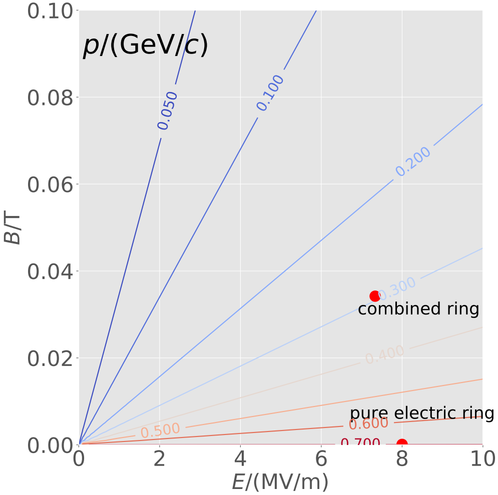
$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{\Omega} \times \vec{s} = \frac{-q}{m}\left[\underbrace{G\vec{B} + \left(G - \frac{1}{\gamma^2 - 1}\right)\vec{v} \times \vec{E}}_{\vec{\Omega}_{\mathrm{MDM}} = 0\,,\quad \textbf{frozen spin}} + \underbrace{\frac{\eta}{2}(\vec{E} + \vec{v} \times \vec{B})}_{= \vec{\Omega}_{\mathrm{EDM}}}\right] \times \vec{s}$$

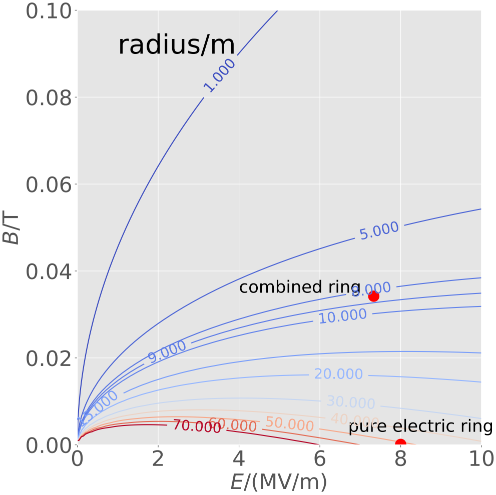
achievable with pure electric field if $G = \frac{1}{\gamma^2 - 1}$, works only for $G > 0$, e.g. proton

or with special combination of E, B fields and γ, i.e. momentum

Momentum and ring radius for **proton** in frozen spin condition

$G = 1.7928474$





Different Options

	☺	☹
3.) pure electric ring	no $\vec{B}$ field needed, $\circlearrowleft$, $\circlearrowright$ beams simultaneously	works only for particles with $G > 0$ (e.g. e, p)
2.) combined ring	works for e, p, d, ^{3}He, smaller ring radius	both $\vec{E}$ and $\vec{B}$ B field reversal for $\circlearrowleft$, $\circlearrowright$ required
1.) pure magnetic ring	existing (upgraded) COSY ring can be used, shorter time scale	lower sensitivity, precession due to G, i.e. no **frozen spin**

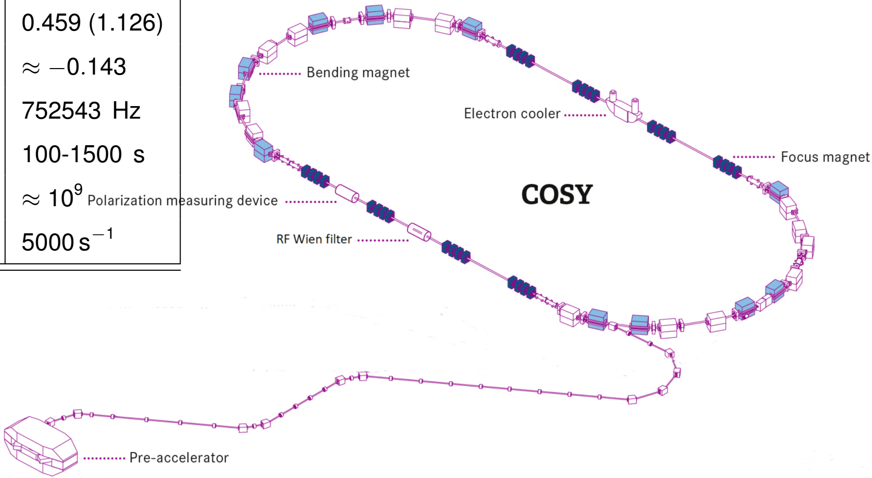
Observable is in all cases a **spin polarization**!

$\rightarrow$ Talk on EDM during workshop

Polarization Measurements

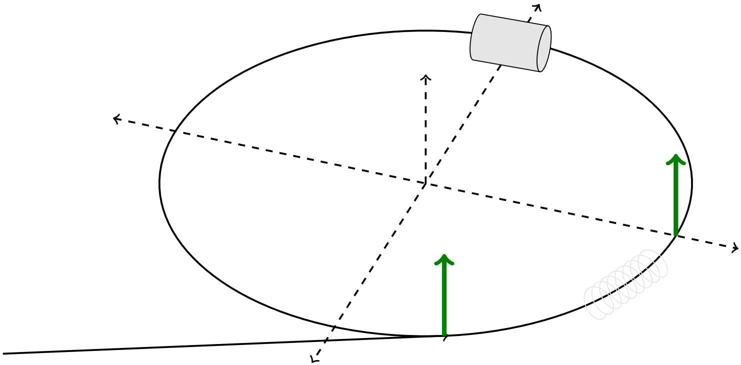
Stage 1: Precursor Experiment

COSY circumference	183 m
deuteron momentum	0.970 GeV/c
$\beta(\gamma)$	0.459 (1.126)
magnetic anomaly G	≈ -0.143
revolution frequency f_{rev}	752543 Hz
cycle length	100-1500 s
nb. of stored particles/cycle	$\approx 10^9$
event rate at $t = 0$	5000 s^{-1}



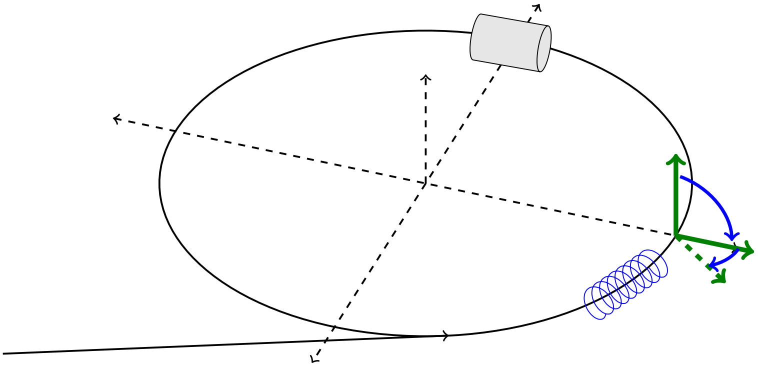
Experimental Setup at COSY

- Inject and accelerate vertically polarized deuterons to $p \approx 1$ GeV/c



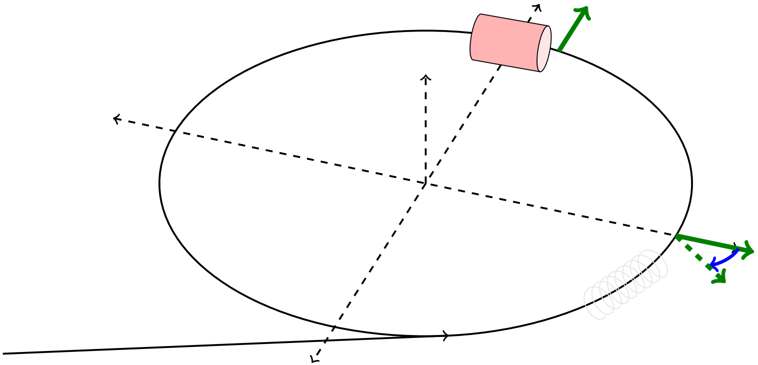
Experimental Setup at COSY

- Inject and accelerate vertically polarized deuterons to $p \approx 1$ GeV/c
- flip polarization with help of solenoid into horizontal plane, precession starts



Experimental Setup at COSY

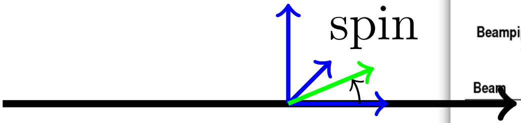
- Inject and accelerate vertically polarized deuterons to $p \approx 1$ GeV/c
- flip polarization with help of solenoid into horizontal plane, precession starts
- Extract beam slowly (in $\approx$ 100-1000 s) on target
- Measure asymmetry and determine spin precession

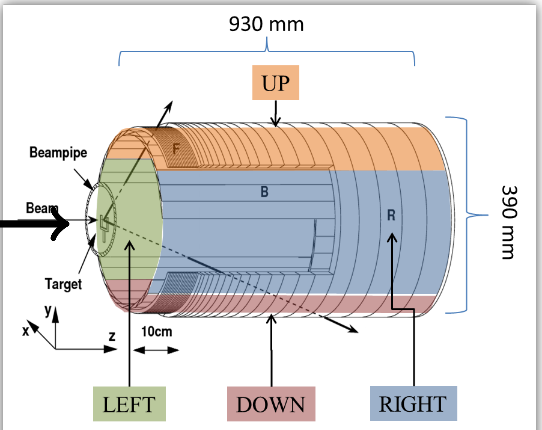


Polarimeter

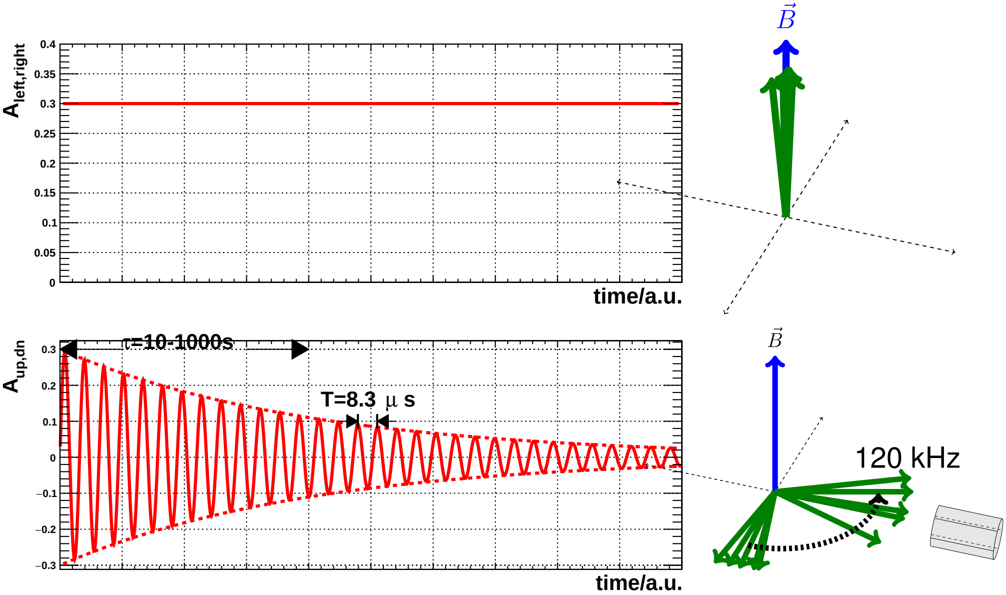
elastic deuteron-carbon scattering,
consists of four scintillator segments: left, right, up, down

asymmetry $A_{up,down} \propto$ horizontal polarization $\rightarrow \nu_s = \gamma G$
asymmetry $A_{left,right} \propto$ vertical polarization $\rightarrow d$

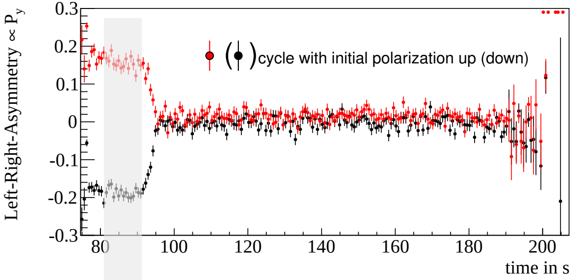




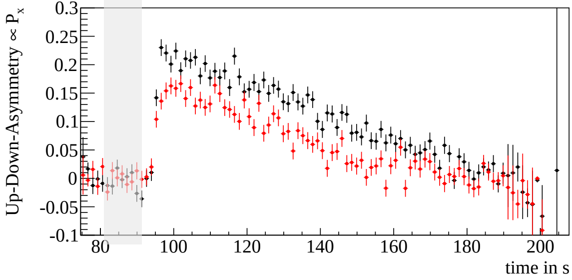
Asymmetries

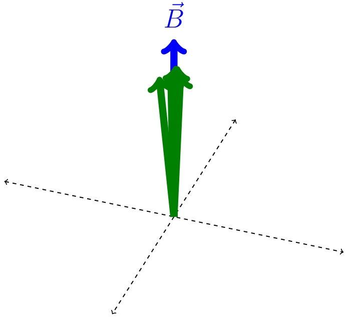


Polarization Flip



envelope of





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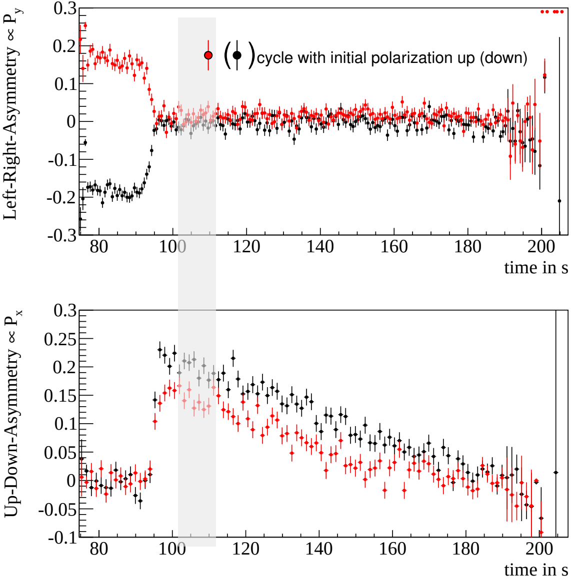

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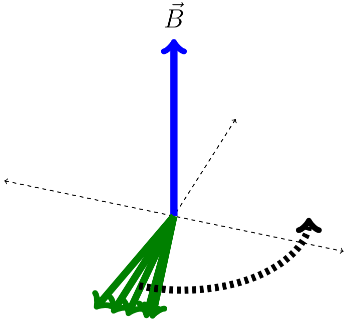

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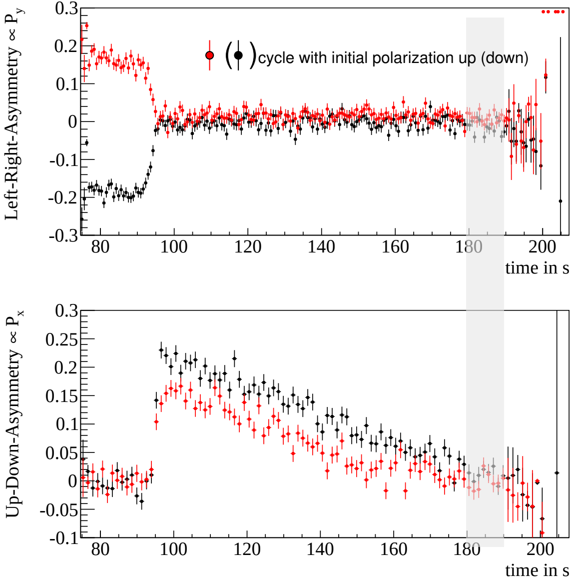
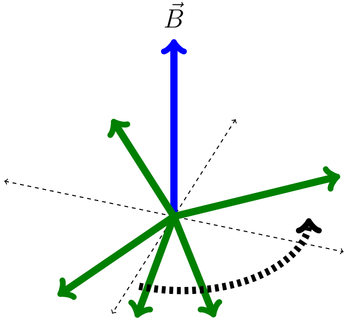
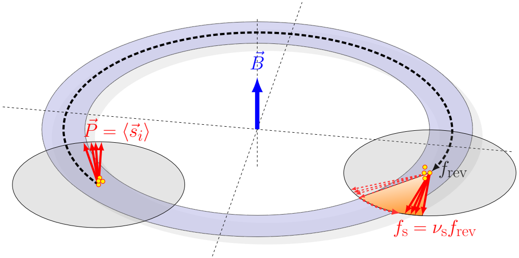
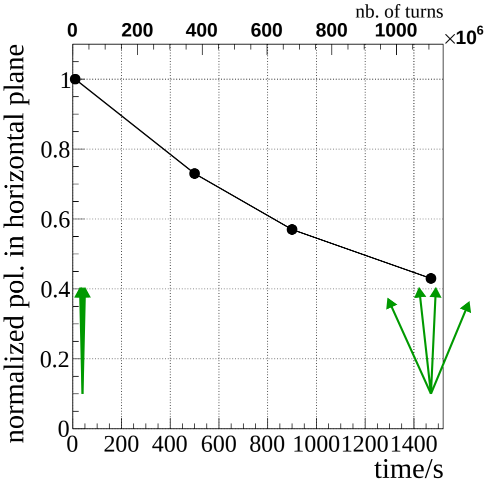




Long Spin Coherence Time (SCT)

Long Spin Coherence time $>$ 1000 s reached





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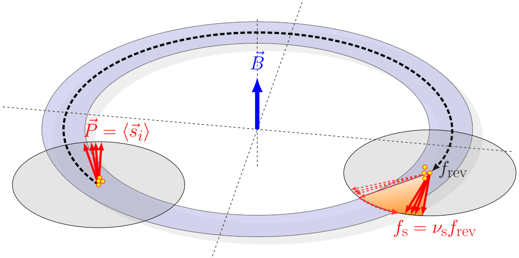
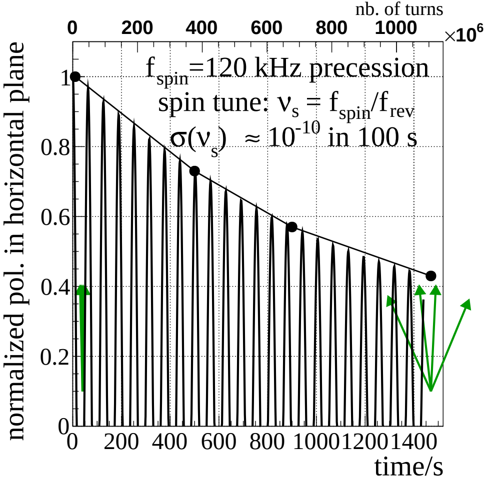
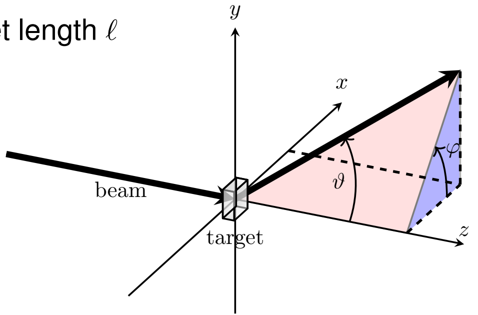




Counting Rates, Cross Section, Polarization

$$N(\vartheta,\varphi) = a(\vartheta,\varphi)\ \mathcal{L}\ \sigma(\vartheta)\ \Big(1 + P\ A(\vartheta)\ \cos(\varphi)\Big)$$

- number of observed events
- acceptance/efficiency
- luminosity
  $\mathcal{L} =$ beam flux $n\times$ target density $\rho\times$ target length ℓ
- unpolarized cross section
- beam polarisation $P = \dfrac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}}$
- analysing power $A = \dfrac{\sigma_L^{\uparrow} - \sigma_R^{\uparrow}}{\sigma_L^{\uparrow} + \sigma_R^{\uparrow}}$



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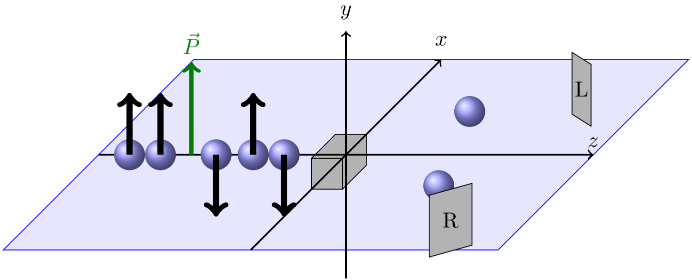


polarisation: $P = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow} = \frac{3-2}{3+2} = 0.2\,,$ analyzing power $A = \frac{\sigma_L^\uparrow - \sigma_R^\uparrow}{\sigma_L^\uparrow + \sigma_R^\uparrow}\,.$

$N_L \propto (n^\uparrow \sigma_L^\uparrow + n^\downarrow \sigma_L^\downarrow)$ Note: $\sigma_L^\uparrow \equiv \sigma_R^\downarrow$

$$\Rightarrow N(\vartheta,\varphi) = \mathcal{L} a(\vartheta,\varphi)\sigma(\vartheta)\Big(1 + P\,A(\vartheta)\,\cos(\varphi)\Big), \qquad \sigma = \frac{1}{2}(\sigma_L + \sigma_R)$$

Counting Rates, Cross Section, Polarization

Derivation of

$$N(\vartheta,\varphi) = a(\vartheta,\varphi)\ \ \mathcal{L}\ \ \sigma(\vartheta)\ \left(1 + P\ \ A(\vartheta)\ \cos(\varphi)\right)$$

Measure counting rates in left and right detector:

$$N(\varphi=0) = N_L \quad \propto \quad n^{\uparrow}\sigma_{\uparrow,L} + n^{\downarrow}\sigma_{\downarrow,L} \stackrel{\varphi-\mathrm{sym}}{=} n^{\uparrow}\sigma_{\uparrow,L} + n^{\downarrow}\sigma_{\uparrow,R}$$

$$N(\varphi=\pi) = N_R \quad \propto \quad n^{\uparrow}\sigma_{\uparrow,R} + n^{\downarrow}\sigma_{\downarrow,R} \stackrel{\varphi-\mathrm{sym}}{=} n^{\uparrow}\sigma_{\uparrow,R} + n^{\downarrow}\sigma_{\uparrow,L}$$

$n^{\uparrow}(n^{\downarrow})$: nb. of beam particles with spin up (down)

$P = \dfrac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}}$: Polarization

$\sigma_{\uparrow,R} \equiv \sigma_R$: cross section for scattering process to the right (R) if spin is up (↑)

$A = \dfrac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$: analyzing power

Connection: Counting rate $\leftrightarrow$ cross section I

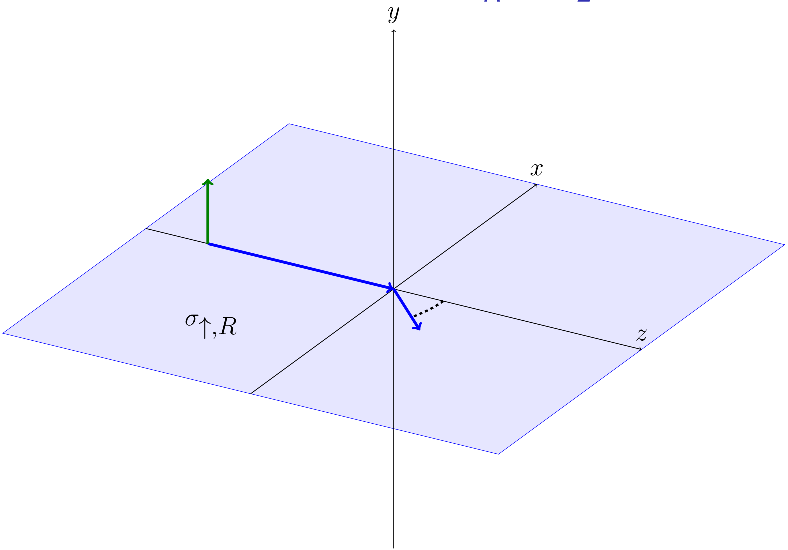
$$\begin{aligned}
N_L &= a\rho\ell\left(n^{\uparrow}\sigma_R + n^{\downarrow}\sigma_L\right) \\
&= a\rho l\frac{1}{2}\Big(n^{\uparrow}\sigma_R + n^{\downarrow}\sigma_L \\
&\qquad + n^{\uparrow}\sigma_R + n^{\downarrow}\sigma_L\Big) \\
&= a\rho\ell\frac{1}{2}\Big(n^{\uparrow}\sigma_R + n^{\uparrow}\sigma_L + n^{\downarrow}\sigma_R + n^{\downarrow}\sigma_L \\
&\qquad + n^{\uparrow}\sigma_R - n^{\uparrow}\sigma_L - n^{\downarrow}\sigma_R + n^{\downarrow}\sigma_L\Big) \\
&= a\rho\ell\frac{1}{2}\left((n^{\uparrow} + n^{\downarrow})(\sigma_R + \sigma_L) + (n^{\uparrow} - n^{\downarrow})(\sigma_R - \sigma_L)\right)
\end{aligned}$$

Connection: Counting rate $\leftrightarrow$ cross section II

$$
\begin{aligned}
N_R &= a\rho\ell \underbrace{\frac{1}{2}(\sigma_R + \sigma_L)}_{=\sigma} \left((n^{\uparrow} + n^{\downarrow}) + (n^{\uparrow} - n^{\downarrow}) \underbrace{\frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L}}_{=A} \right) \\
&= a\rho\ell \underbrace{(n^{\uparrow} + n^{\downarrow})}_{=n} \sigma \left(1 + \underbrace{\frac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}}}_{=P} A \right) \\
&= a\mathcal{L}\sigma \,(1 + PA) \\
N_R &= a\mathcal{L}\sigma \,(1 - PA)
\end{aligned}
$$

P : Polarization(to be determined), A : analyzing power(known)

$$\sigma_R^{\uparrow} = \sigma_L^{\downarrow}$$



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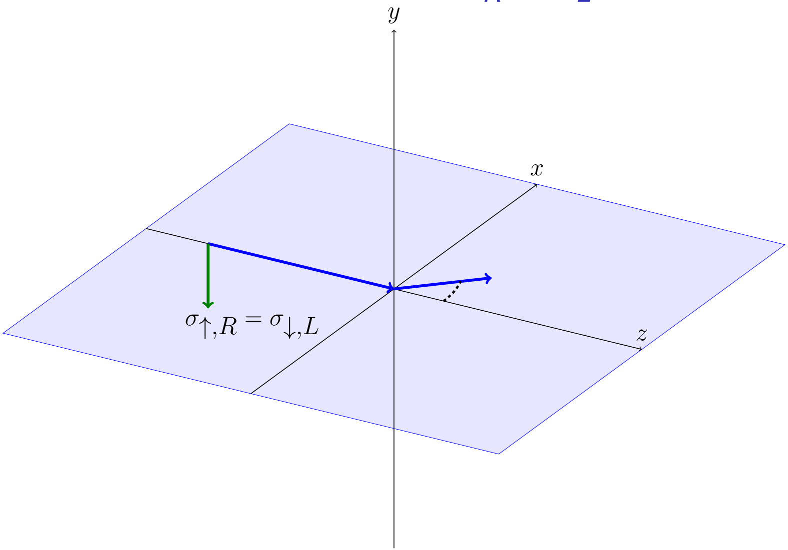
$\sigma_R^{\uparrow} = \sigma_L^{\downarrow}$



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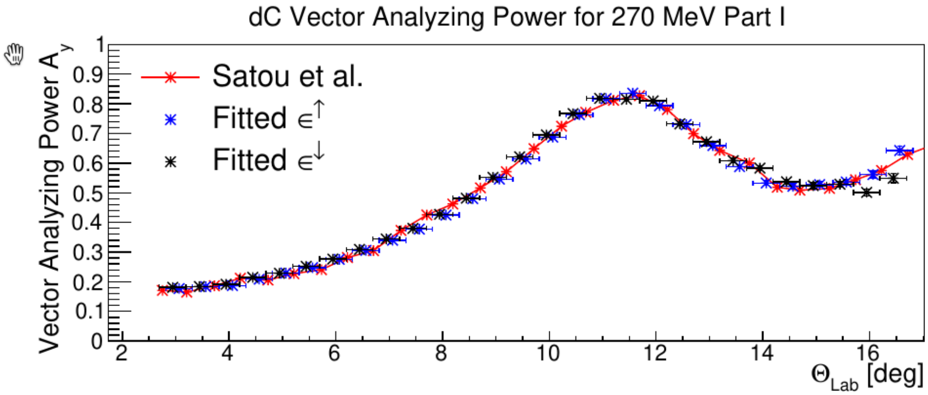
Example for analysing power

deuteron carbon scattering, $p = 970\text{MeV}/c$



, PhD Thesis F. Müller, RWTH Aachen, 2019

Goal for EDM measurements

Determine P from counting rate $N(\vartheta, \varphi)$ and analysing power $A(\vartheta)$ with small uncertainty σ_P (without knowing $\mathcal{L}$, a and σ).

To simplify the discussion

- assume constant acceptance in φ: $\dfrac{\partial a(\vartheta, \varphi)}{\partial \varphi} = 0$
- detector placed at one polar angle ϑ

We are left with

$$N(\varphi) = \frac{1}{2\pi} N_0 (1 + PA \cos(\varphi)) \quad , N_0 = a\mathcal{L}\sigma$$

Most easy way to get P

Just consider counts in the left part of the detector $\varphi \approx 0, \cos(\varphi) = 1$ and the right part $\varphi \approx \pi, \cos(\varphi) = -1$.

$$
\begin{aligned}
\langle N_L \rangle &= N_0 \frac{\Delta\varphi}{2\pi}(1 + AP) \\
\langle N_R \rangle &= N_0 \frac{\Delta\varphi}{2\pi}(1 - AP)
\end{aligned}
$$

Consider a **counting rate asymmetry**

$$\hat{P} = \frac{1}{A} \frac{N_L - N_R}{N_L + N_R} ,$$

$\hat{P}$: estimator for P.

If A is known, one can determine P.

Note:

$\langle N_{L,R} \rangle$: expectation value

$N_{L,R}$: actually measured number of events

What about the error?

Error propagation gives: $\sigma_P = \dfrac{1}{A\sqrt{N}}$

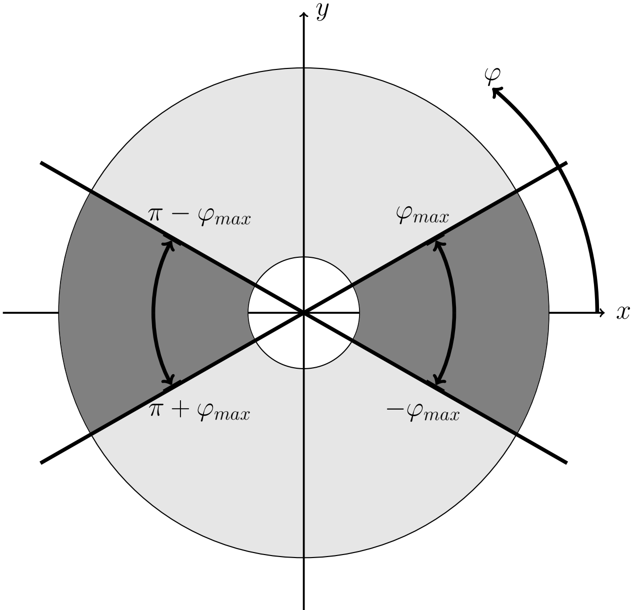
(assuming $PA \ll 1$, i.e. $N_L \approx N_R =: N/2$)
As in any counting experiment the statistical error scales with $1/\sqrt{N}$.
Counting only events in small region $\Delta\varphi$ around $\varphi = 0$ and π results in small $N = N_0 \dfrac{2\Delta\varphi}{2\pi}$ and thus large error.

It's more convenient to work with the Figure of Merit (FOM):

$\mathrm{FOM}_P = \sigma_P^{-2} = N\,A^2$

How does error change if we include more events, i.e. making $\Delta\varphi$ larger?

Enlarge φ range



Enlarge φ range

estimator $\hat{P} = \frac{1}{A\langle\cos(\varphi)\rangle}\frac{N_L - N_R}{N_L + N_R}$

$$\sigma_P = \frac{1}{\sqrt{N}}\frac{1}{A\langle\cos(\varphi)\rangle},$$

number of events: $N = \frac{4\varphi_{max}}{2\pi}$

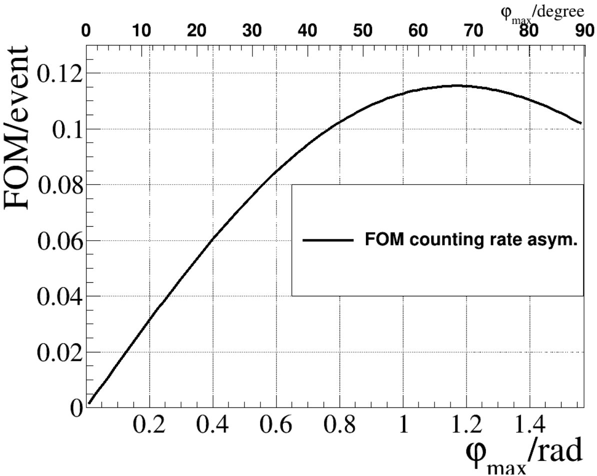
$$\varphi_{max} \nearrow \quad \Rightarrow \quad N \nearrow$$

$$\varphi_{max} \nearrow \quad \Rightarrow \quad \langle\cos(\varphi)\rangle \searrow$$

$$\langle\cos(\varphi)\rangle = \frac{\int_{-\varphi_{max}}^{\varphi_{max}} \cos(\varphi)\mathrm{d}\varphi}{2\varphi_{max}}$$

$$\mathrm{FOM}_P = \sigma_P^{-2} = N\left(A\langle\cos(\varphi)\rangle\right)^2$$

Figure of Merit (FOM)



- strange behavior: Adding data beyond $\varphi_{max} > 67^{\circ}$ the FOM decreases
- Reason: adding data at larger φ "dilutes" the sample

Can one do better? Yes! **Event Weighting**

Instead of just counting events, weight every event with a weight function $w(\varphi)$.

Estimator for P

$$\hat{P} = \frac{1}{A} \frac{\sum_{L,R} w_i}{\sum_{L,R} w_i \cos(\varphi_i)}$$

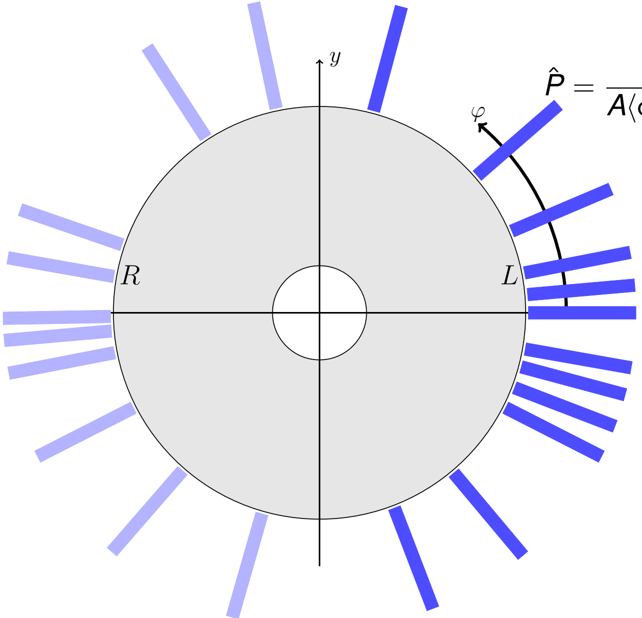
In principle weight w arbitrary, two cases are of interest

- $w = 1$ (left), $w = -1$ (right): $\hat{P} = \frac{1}{A\langle\cos(\varphi)\rangle} \frac{N_L - N_R}{N_L + N_R}$ (counting rate asymmetry)
- $w = A\cos(\varphi)$: $\hat{P} = \frac{1}{A} \frac{\sum_{L,R} \cos(\varphi_i)}{\sum_{L,R} \cos^2(\varphi_i)}$ (optimal weight)[1]

choice $w(\varphi) \equiv A\cos(\varphi)$ leads to smallest statistical error.

[1] In terms of highest FOM.

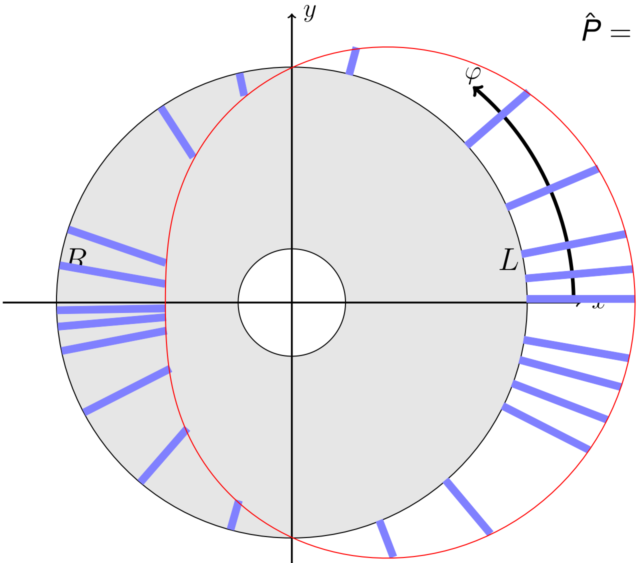
Every event weighted with $w = 1$



$$\hat{P} = \frac{1}{A\langle\cos(\varphi)\rangle} \frac{N_L - N_R}{N_L + N_R}$$

Every event weighted with $w = A\cos(\varphi)$

$$\hat{P} = \frac{1}{A} \frac{\sum \cos(\varphi_i)}{\sum \cos^2(\varphi_i)}$$



What about the error?

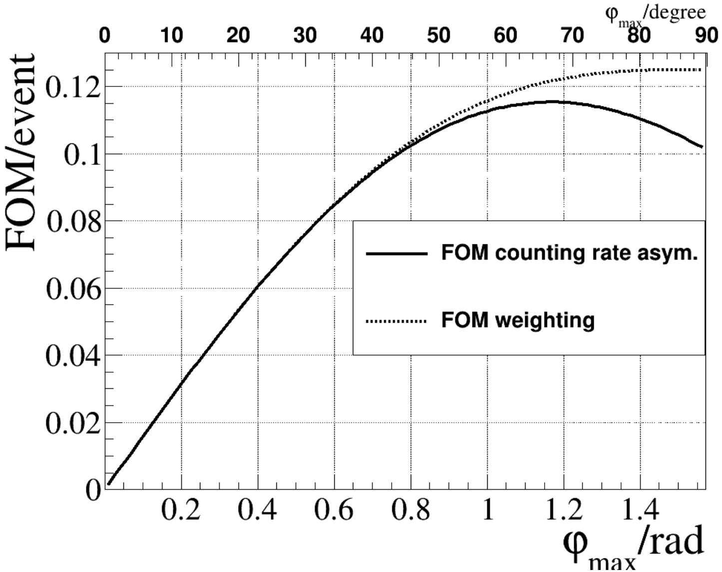
Error Propagation: $\text{FOM}_P = NA^2 \frac{\langle w \cos(\varphi) \rangle^2}{\langle w^2 \rangle}$

	counting, $w = 1$	$w = A\cos(\varphi)$, MLH, binning
FOM_P	$NA^2 \langle \cos(\varphi) \rangle^2$	$NA^2 \langle \cos(\varphi)^2 \rangle$

Gain in FOM: $\frac{\langle \cos(\varphi)^2 \rangle}{\langle \cos(\varphi) \rangle^2} \geq 1$

An event with a large $\cos(\varphi)$ tells you more about P than an event with lower $\cos(\varphi)$. It should thus enter the analysis with more weight.

FOM



φmax/degree
0
10
20
30
40
50
60
70
80
90
FOM/event
0.12
0.1
0.08
0.06
0.04
0.02
0
0.2
0.4
0.6
0.8
1
1.2
1.4
φmax/rad
FOM counting rate asym.
FOM weighting

Connection to Maximum Likelihood Method

$N(\varphi) \propto (1 + A\cos(\varphi)P) = (1 + \beta(\varphi)P)\,,$
Here: $\beta(\varphi) = A\cos(\varphi)$

Log-likelihood function

$$\ell \;=\; \sum_{i=1}^{N} \ln\left(1 + \beta(\varphi_i)P\right)$$

Connection to Maximum Likelihood Method

MLH estimator for P: Maximize $\ell \Rightarrow \frac{\partial \ell}{\partial P} \overset{!}{=} 0$

$$\Rightarrow \frac{\partial \ell}{\partial P} = \sum_i \frac{\beta(\varphi_i)}{1 + \beta(\varphi_i) P} = 0$$

for $\beta(\varphi_i) P \ll 1$:

$$\Rightarrow \sum_i \beta(\varphi_i)(1 - \beta(\varphi_i) P) = 0$$

$$\Rightarrow \hat{P} = \frac{\sum_i \beta(\varphi_i)}{\sum_i \beta^2(\varphi_i)} = \frac{1}{A} \frac{\sum_i \cos(\varphi_i)}{\sum_i \cos^2(\varphi_i)}$$

Estimator of maximum likelihood function coincides with estimator for optimal weight!

It is well known that MLH estimator reach largest FOM (Cramer-Rao-bound).

More general case

events follow distribution $N(\vec{x}) \propto (1 + \beta(\vec{x})P)$

For optimal event weight/MLH FOM is given by

$$\mathrm{FOM}_P = N\langle\beta(\vec{x})^2\rangle$$

Counting rates reach only

$$\mathrm{FOM}_P = N\langle\beta(\vec{x})\rangle^2$$

$$\langle\beta(\vec{x})\rangle = \frac{\int_X \beta(\vec{x})\mathrm{dx^n}}{\int_X \mathrm{dx^n}}, \quad X = \text{acc. events}$$

for example $\beta(\vec{x}) = \beta(\vartheta, \varphi) = A(\vartheta)\cos(\varphi)$

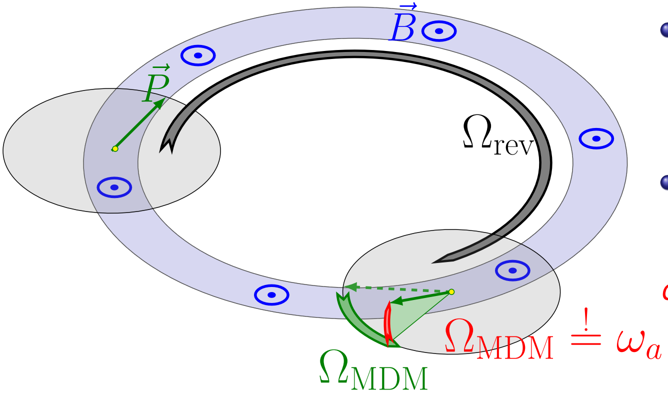
Summary

- Polarizations can be extracted from azimuthal dependent event rates, knowing the analyzing power A
- weighting the events with $\cos(\varphi)$ give the largest FOM
- Gain with respect to just counting events is $\frac{\mathrm{FOM}_{w=A\cos(\varphi)}}{\mathrm{FOM}_{cnt}} = \frac{\langle\cos(\varphi)^2\rangle}{\langle\cos(\varphi)\rangle^2}$
- Assumption made on acceptance, $PA \ll 1$, fixed ϑ, ... were only made to simplify discussions

More details in [1], [2] [3],[4], [5], [6], [7]

Axion searches at Storage Rings

Principle of storage ring axion experiment



- Axion field gives rise to an effective time-dependent θ-QCD term
- This gives rise to an oscillating electric dipole moment EDM d.

$$d = d_{DC} + d_{AC} \sin(\omega_a t + \varphi_a)$$

$$\omega_a = \frac{m_a c^2}{\hbar}$$

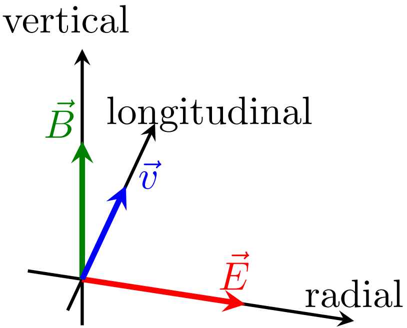
Derive analytic expressions for spin motion with oscillating EDM

Starting Point: BMT-Equation I

Equation of motion in matrix form

$$\frac{d\vec{S}}{dt} = (A_{\mathrm{MDM}} + \eta A_{\mathrm{EDM}})\vec{S} \quad (1)$$

$$\vec{S} = (S_r, S_v, S_\ell)$$



with

$$A_{\mathrm{MDM}} = \begin{pmatrix} 0 & 0 & \Omega_{\mathrm{MDM}} \\ 0 & 0 & 0 \\ -\Omega_{\mathrm{MDM}} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta A_{\mathrm{EDM}} = \eta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Omega_{\mathrm{EDM}} \\ 0 & \Omega_{\mathrm{EDM}} & 0 \end{pmatrix}.$$

Starting Point: BMT-Equation II

$$\Omega_{\mathrm{MDM}} = -\frac{q}{m}\left(GB + \left(G - \frac{1}{\gamma^2 - 1}\right)\frac{\beta E}{c}\right), \quad \Omega_{\mathrm{EDM}} = -\frac{q}{2mc}(E + c\beta B).$$

In the following we assume that the EDM can have a constant term and a time varying component, thus $\eta = \eta_0 + \eta_1 \cos(\omega_a t + \varphi_a)$

Note relationship between dimensionless parametet η and EDM d:
$\vec{d} = \eta \frac{q\hbar}{2mc}\vec{s}$

Since $\eta_0, \eta_1 \ll G$, A_{EDM} can be treated as an perturbation.

Solution I

Again equation of motion:

$$\dot{\vec{S}} = (A_{\mathrm{MDM}} + \eta A_{\mathrm{EDM}}(t))\vec{S}\,. \tag{2}$$

To solve equation 2 we expand the solution in orders of η

$$\vec{S}(t) = \vec{S}_0(t) + \eta \vec{S}_1(t) \tag{3}$$

Entering equation 3 in equation 2 and keeping only terms up to order one in η yields

$$\dot{\vec{S}}_0 + \eta \dot{\vec{S}}_1 = A_{\mathrm{MDM}}\vec{S}_0 + \eta(A_{\mathrm{MDM}}\vec{S}_1 + A_{\mathrm{EDM}}\vec{S}_0)\,. \tag{4}$$

Thus

$$\dot{\vec{S}}_0 = A_{\mathrm{MDM}}\vec{S}_0\,, \tag{5}$$

$$\dot{\vec{S}}_1 = (A_{\mathrm{MDM}}\vec{S}_1 + A_{\mathrm{EDM}}\vec{S}_0)\,. \tag{6}$$

Solution II

Since A_{MDM} does not depend on t, equation 5 has the solution

$$\vec{S}_0(t) = \exp(A_{\mathrm{MDM}} t)\vec{S}(0) \tag{7}$$

with arbitrary initial condition $\vec{S}(0)$.

The solution for the equation 6 is:

$$\vec{S}_1(t) = \exp(A_{\mathrm{MDM}} t)\vec{S}(0) + \int_0^t \exp(A_{\mathrm{MDM}}(t-s))A_{\mathrm{EDM}}\vec{S}_0(s)\mathrm{d}s\,. \tag{8}$$

(Duhamel's formula)

Test: Prove that ansatz 8 solves eq. 6

$$\vec{S}_1(t) = \exp(A_{\mathrm{MDM}} t)\vec{S}(0) + \exp(A_{\mathrm{MDM}} t) \int_0^t \exp(-A_{\mathrm{MDM}} s) A_{\mathrm{EDM}} \vec{S}_0(s) \mathrm{d}s \qquad (9)$$

$$\begin{aligned}
\frac{dS_1}{dt} &= A_{\mathrm{MDM}} \exp(A_{\mathrm{MDM}} t)\vec{S}(0) + \\
&\quad A_{\mathrm{MDM}} \exp(A_{\mathrm{MDM}} t) \int_0^t \exp(-A_{\mathrm{MDM}} s) A_{\mathrm{EDM}} \vec{S}_0(s) \mathrm{d}s + \\
&\quad \exp(A_{\mathrm{MDM}} t) \exp(-A_{\mathrm{MDM}} t) A_{\mathrm{EDM}} \vec{S}_0(t) \\
&= A_{\mathrm{MDM}} \left(\exp(A_{\mathrm{MDM}} t)\vec{S}(0) + \exp(A_{\mathrm{MDM}} t) \int_0^t \exp(-A_{\mathrm{MDM}} s) A_{\mathrm{EDM}} \vec{S}_0(s) \mathrm{d}s \right) + A_{\mathrm{EDM}} \\
&= A_{\mathrm{MDM}} \vec{S}_1 + A_{\mathrm{EDM}} \vec{S}_0
\end{aligned}$$

Solution

Up to first order in η the solution is

$$\begin{aligned}\vec{S}(t) &= \vec{S}_0(t) + \eta \vec{S}_1(t) \\ &= (1+\eta)\exp(A_{\mathrm{MDM}} t)\,\vec{S}(0) + \eta \int_0^t \exp(A_{\mathrm{MDM}}(t-s)) A_{\mathrm{EDM}} \exp(A_{\mathrm{MDM}} s)\,\vec{S}(0)\mathrm{d}s\end{aligned}$$

Solution (1st order in η_0 and η_1)

Vertical component $S_v(t)$ for initial condition $\vec{S}(0) = (0, 0, 1)$:

$$
\begin{aligned}
S_v(t) \quad = \quad & \eta_0 \Omega_{\mathrm{EDM}} \frac{\sin(\Omega_{\mathrm{MDM}} t)}{\Omega_{\mathrm{MDM}}} + \eta_1 \frac{\Omega_{\mathrm{EDM}}}{2(\Omega_{\mathrm{MDM}} - \omega_a)(\Omega_{\mathrm{MDM}} + \omega_a)} \\
& \Big[-2\omega_a \sin(\varphi_a) \\
& + (\omega_a + \Omega_{\mathrm{MDM}}) \sin((\Omega_{\mathrm{MDM}} - \omega_a)t + \varphi_a) \\
& + (\omega_a - \Omega_{\mathrm{MDM}}) \sin((\Omega_{\mathrm{MDM}} + \omega_a)t + \varphi_a) \Big]
\end{aligned}
$$

looks complicated but close to resonance:
$\Omega_{\mathrm{MDM}} + \omega_a \gg \Omega_{\mathrm{MDM}} - \omega_a$
details: [8]

Solution: Special Cases

Ignore all fast oscillating terms and setting $\varphi_a = 0$:

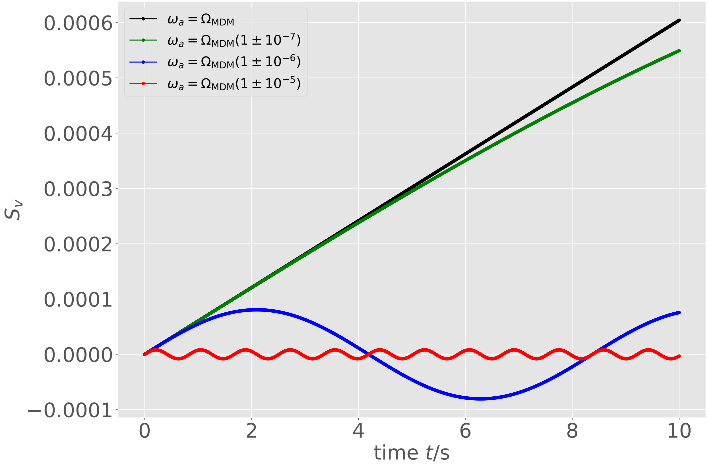
$$S_v(t) = \eta_1 \frac{\Omega_{\mathrm{EDM}}}{2(\omega_a - \Omega_{\mathrm{MDM}})} \sin\left((\Omega_{\mathrm{MDM}} - \omega_a)t\right) .$$

In resonance ($\omega_a = \Omega_{\mathrm{MDM}}$):

$$S_v(t) = \eta_1 \frac{\Omega_{\mathrm{EDM}}}{2} t .$$

Largest build-up (i.e. smallest error on η_1) for $\omega_a = \Omega_{\mathrm{MDM}}$ (and $\varphi_a = 0$).

Vertical polarisation S_v vs. time t



looks much simpler than formula on previous page,

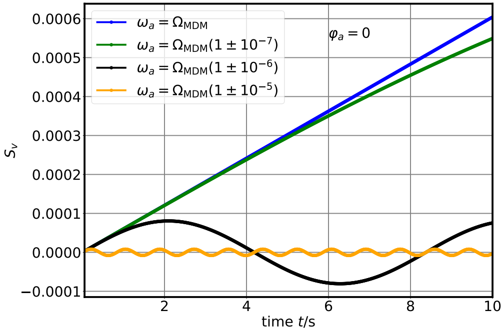
reason: fast oscillating terms can be ignored

$\Omega_{\mathrm{MDM}} = 750000.0\,\mathrm{s}^{-1}$, $\Omega_{\mathrm{EDM}} = 1208341\,\mathrm{s}^{-1}$, $\eta_0 = 0$, $\eta_1 = 10^{-10}$ and $\varphi_a = 0$.

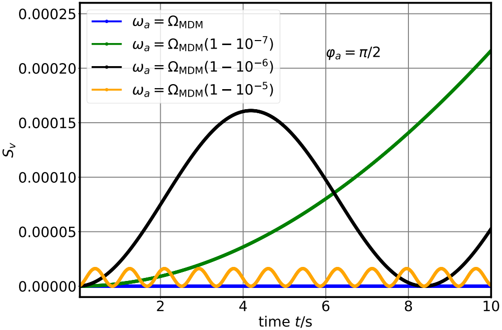
Meaning of phase φ_a

Axion field is given by $\propto \sin(\omega_a t + \varphi_a)$,
but phase φ_a is not known:

$\varphi_a = 0$

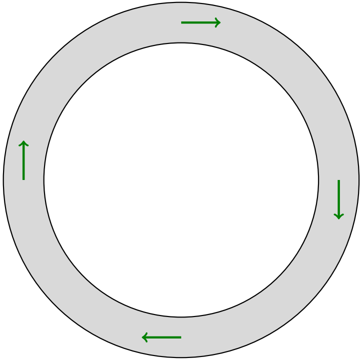


$\varphi_a = \pi/2$



How to assure that $\varphi_a = 0$

φ_a: phase between axion field and spin vector
$\rightarrow$ put four bunches in ring with different orientation of spin
$\rightarrow$ There are always two bunches which will pick up the axion signal.



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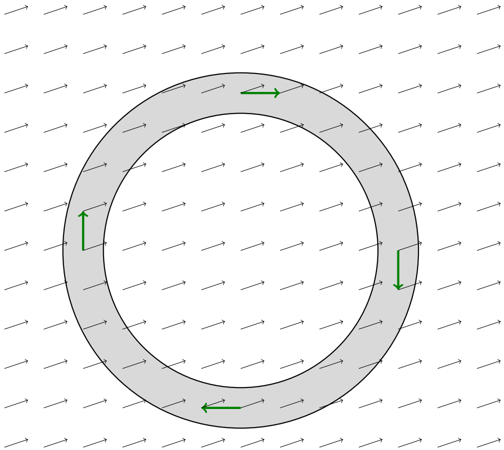
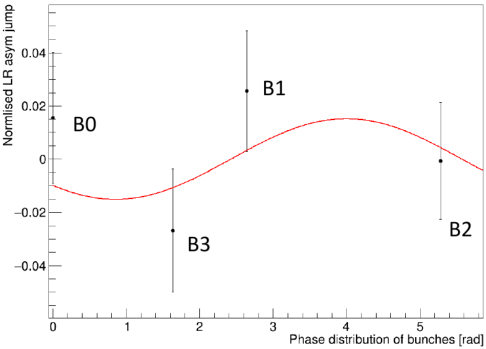


Asymmetry for one frequency Ω_{MDM}



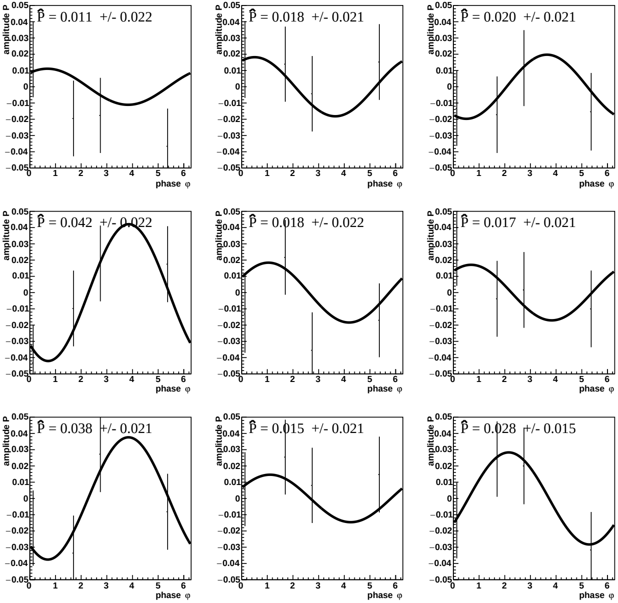
LR-asymmetry for 4 bunches

on average:

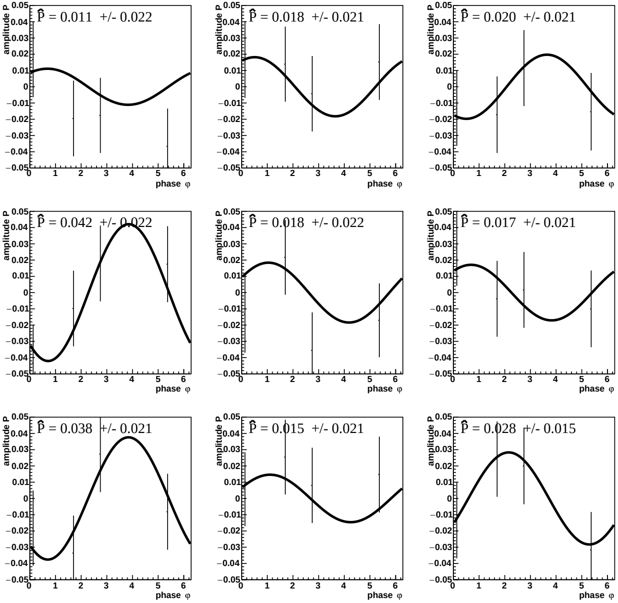
$\hat{P} = 0.023 \pm 0.020,$

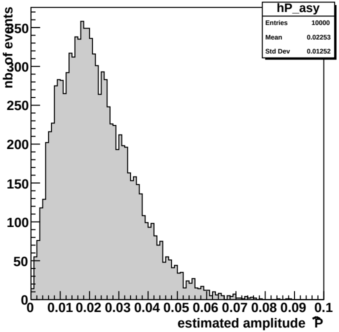
Axion would show up as a non-zero amplitude

Example from MC simulation



Example from MC simulation





true amplitude $P = 0$
but recontructed amplitude
$\hat{P} = 0.023 \pm 0.020$

Fitting function

- Axion signal shows up in a non-zero amplitude of sin wave.
- If amplitude close to 0, there is a well known bias in determining the amplitude.

Fitting function:

$$f(\varphi; A, B) = A\sin(\varphi) + B\cos(\varphi)$$

gives estimates for $\hat{A}$ and $\hat{B}$ for parameters A and B.

Estimate for amplitude P: $\hat{P} = \sqrt{\hat{A}^2 + \hat{B}^2} \geq 0$

Analytic expression for pdf $f(\hat{P}|P)$

If $\hat{A}$ and $\hat{B}$ are are uncorrelated and normal distributed with means A and B:

$$f(\hat{A}|A)f(\hat{B}|B)\mathrm{d}\hat{A}\mathrm{d}\hat{B} = \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-(\hat{A}-A)^2/(2\sigma^2)}\mathrm{e}^{-(\hat{B}-B)^2/(2\sigma^2)}\,\mathrm{d}\hat{A}\mathrm{d}\hat{B}$$

$\hat{P} = \sqrt{\hat{A}^2 + \hat{B}^2}$ follows

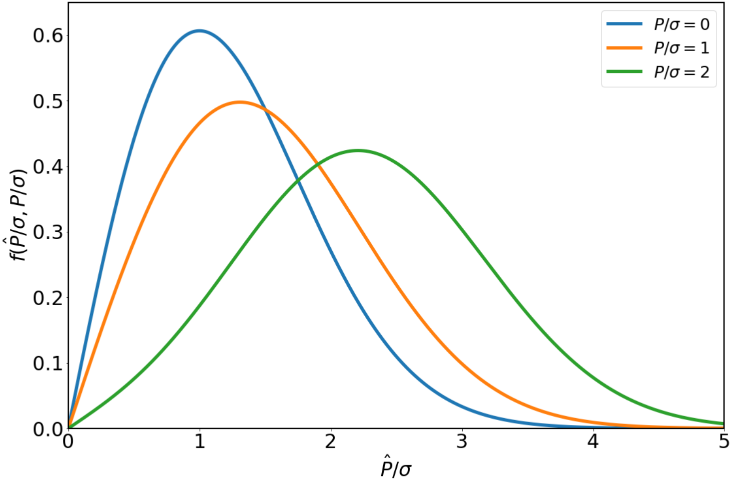
$$f(\hat{P}|P)\mathrm{d}\hat{P} = \frac{1}{\sigma^2}\,\mathrm{e}^{-(\hat{P}^2+P^2)/(2\sigma^2)}\,\hat{P}\,I_0\left(\frac{\hat{P}P}{\sigma^2}\right)\,\mathrm{d}\hat{P}$$ Rice distribution

where I_0 is the modified Bessel function of first kind, σ is error on $\hat{A}$,$\hat{B}$ and also $\hat{P}$.

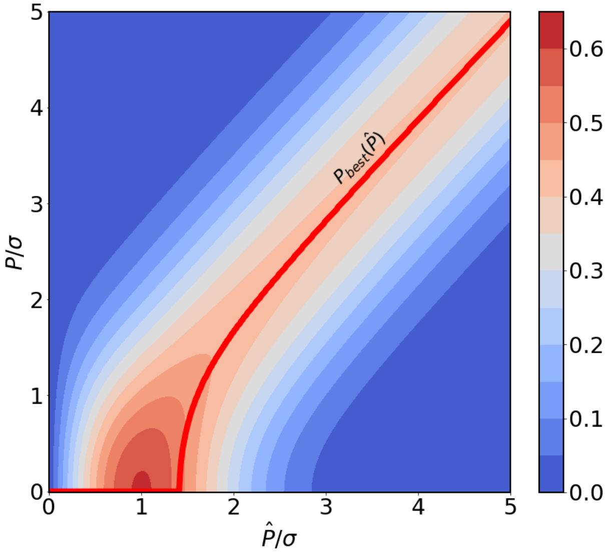
Can be written in terms of $\epsilon = P/\sigma$

$$f\left(\hat{\epsilon}|\epsilon\right)\mathrm{d}\hat{\epsilon} = \mathrm{e}^{-\left(\hat{\epsilon}^2+\epsilon^2\right)}\,\hat{\epsilon}\,I_0\left(\hat{\epsilon}\epsilon\right)\,\mathrm{d}\hat{\epsilon}$$

$f(\hat{\epsilon}, \epsilon)$



$f(\hat{\epsilon}, \epsilon)$



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Feldman-Cousins Confidence Interval

Simple least square fit gives estimate $\hat{P} \pm \sigma$ which has

- a bias
- may have coverage for $P < 0$. (i.e. 2σ interval: $0.023 \pm 2 \cdot 0.020$)

How to get confidence interval with coverage only for $P > 0$?
$\Rightarrow$ Feldman-Cousins confidence interval [9]

Feldman-Cousins Confidence Interval

At a given value of true P include all values of $\hat{P}$ in the confidence interval for which the ratio

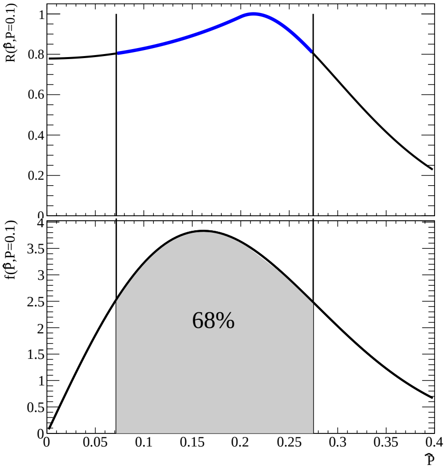
$$R(\hat{P}, P) = \frac{f(\hat{P}|P)}{f(\hat{P}|P_{\text{best}})}$$

has the largest values until the desired coverage of the confidence interval is reached.

P_{best} denotes the value for which $f(\hat{P}|P_{\text{best}})$ has its maximum in the allowed region of P, i.e. $f(\hat{P}|P_{\text{best}}) = \max\{f(\hat{P}|P), P > 0\}$.

This is done without looking at the data.

Construction for $P = 0.1$ and $N = 100$ (i.e. $\sigma = \sqrt{\frac{2}{100}} \approx 0.14$)

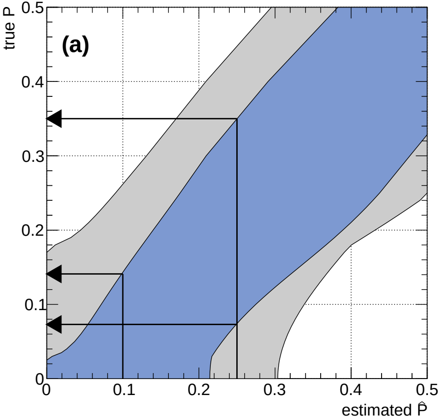


$$R(\hat{P}, P) = \frac{f(\hat{P}|P)}{f(\hat{P}|P_{\text{best}})}$$

$$f(\hat{P}, P = 0.1)$$

Do this for all values of P ...

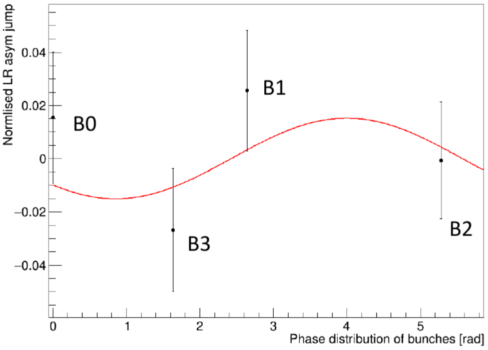
Confidence Interval



- construct horizontally
- read off vertically
  Ex.1: if $\hat{P} = 0.1$, the 68% CI (blue area) for P: [0,0.14]
  Ex.2: if $\hat{P} = 0.25$, the 68% CI for P: [0.075,0.35]
- if $P, \hat{P} \gg \sigma$ normal Gaussian intervals are obtained
- no arbitrary choice of two sided or one-sided (i.e. upper limit) interval

Details in [10, 11]

Back to data

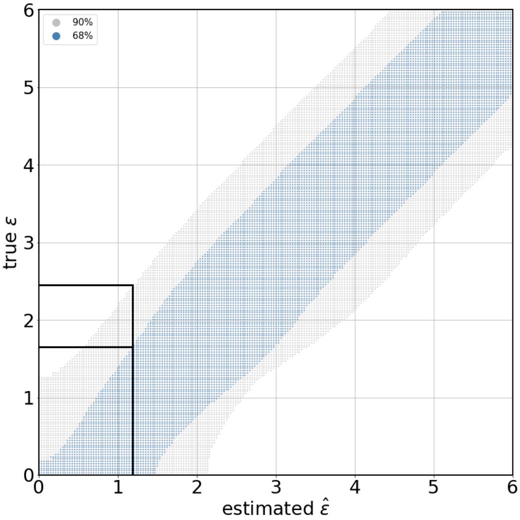


LR-asymmetry for 4 bunches

on average:

$\hat{P} = 0.023 \pm 0.020,$

Confidence Limit 1 cycle

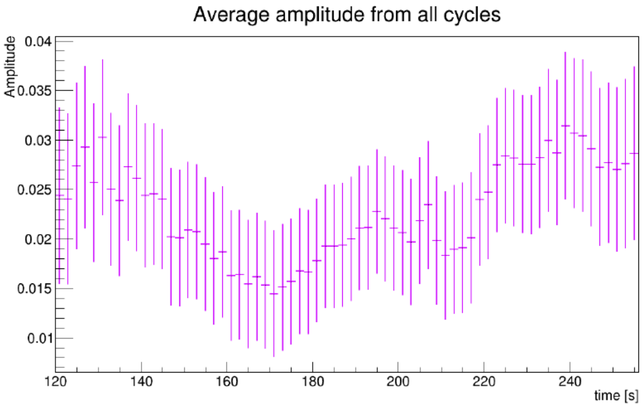


$$\epsilon = \frac{P}{\sigma}, \quad \hat{\epsilon} = \frac{\hat{P}}{\sigma}$$

result:
$\hat{P} = 0.023 \pm 0.020$,
i.e. $\frac{\hat{P}}{\sigma} \approx 1.2$

68% CI = [0,1.6]
90% CI = [0,2.4]

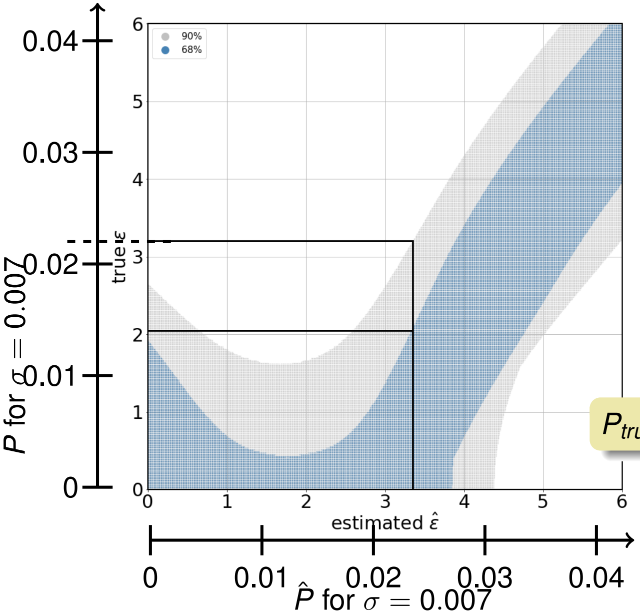
Amplitude for many frequencies and **8 cycles**



From plot one can get impression that we observe a non-zero amplitude (8 cycles combined),

result:
$\hat{P} = 0.023 \pm 0.007$,
i.e. $\frac{\hat{P}}{\sigma} \approx 3.3$

Confidence Intervals **8 cycles**



observed 0.023 ± 0.007

(see page 13)

i.e. ${}^{0.023}/_{0.007} = 3.3$

$\Rightarrow$ 90% CL: 3.1 = ϵ_{true}/σ, i.e

P_{true} in $[0, 0.022]$ 90% CI

Note: Special treatment needed if $\epsilon_{estimated}/\sigma < 3.3$

Summary

- axions lead to oscillating EDM
- signal is amplitude of sine signal
- bias if amplitude is close to zero
- algorithm based on Feldman-Cousins method gives correct confidence limit

Literature I

[11]

[1] F. Müller et al., "Measurement of deuteron carbon vector analyzing powers in the kinetic energy range 170-380 MeV," Eur. Phys. J. A, vol. 56, no. 8, p. 211, 2020.

[2] Pretz, J. and Müller, F., "Extraction of Azimuthal Asymmetries using Optimal Observables," Eur. Phys. J., vol. C79, no. 1, p. 47, 2019.

[3] C. Adolph et al., "Longitudinal double spin asymmetries in single hadron quasi-real photoproduction at high p_T," Phys. Lett., vol. B753, pp. 573–579, 2016.

[4] M. Alekseev et al., "Gluon polarisation in the nucleon and longitudinal double spin asymmetries from open charm muoproduction," Phys. Lett., vol. B676, pp. 31–38, 2009.

Literature II

[5] G. Bennett et al., "Measurement of the negative muon anomalous magnetic moment to 0.7 ppm," Phys. Rev. Lett., vol. 92, p. 161802, 2004.

[6] J. Pretz, "Comparison of methods to extract an asymmetry parameter from data," Nucl. Instrum. Meth., vol. A659, pp. 456–461, 2011.

[7] J. Pretz and J.-M. Le Goff, "Simultaneous Determination of Signal and Background Asymmetries," Nucl. Instrum. Meth., vol. A602, pp. 594–596, 2009.

[8] J. Pretz, S. Karanth, E. Stephenson, S. P. Chang, V. Hejny, S. Park, Y. Semertzidis, and H. Ströher, "Statistical sensitivity estimates for oscillating electric dipole moment measurements in storage rings," Eur. Phys. J. C, vol. 80, no. 2, p. 107, 2020.

Literature III

[9] G. J. Feldman and R. D. Cousins, “A Unified approach to the classical statistical analysis of small signals,” Phys. Rev., vol. D57, pp. 3873–3889, 1998.

[10] S. Plaszczynski, L. Montier, F. Levrier, and M. Tristram, “A novel estimator of the polarization amplitude from normally distributed Stokes parameters,” Monthly Notices of the Royal Astronomical Society, vol. 439, pp. 4048–4056, 02 2014.

[11] D. Eversmann, J. Pretz, and M. Rosenthal, “Amplitude estimation of a sine function based on confidence intervals and Bayes’ theorem,” JINST, vol. 11, no. 05, p. P05003, 2016.