

Jülich-Bonn-Washington model for pion electroproduction multipoles

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Pion electroproduction off the proton is analyzed in a new framework based on a general parametrization of transition amplitudes, including constraints from gauge invariance and threshold behavior. Data with energies $1.13 \text{ GeV} < W < 1.6 \text{ GeV}$ and Q^2 below 6 GeV^2 are included. The model is an extension of the latest Jülich-Bonn solution incorporating constraints from pion-induced and photoproduction data. Performing large-scale fits ($\approx 10^5$ data) we find a set of solutions with $\chi^2_{\text{dof}} = 1.69\text{--}1.81$ which allows us to assess the systematic uncertainty of the approach.

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I. INTRODUCTION

Our knowledge of the baryon spectrum, as determined from analyses of data, has advanced rapidly [1] over the past decade. The progress has been most significant for nonstrange baryons, largely due to the wealth of new and more precise measurements made at electron accelerators worldwide. A substantial number of these new measurements have been performed at Jefferson Lab (JLab) [2] using the CLAS and Hall A detectors, at MAMI [3], at ELSA [4] with the Crystal Barrel detector, and also at the BESIII [5] and LEPS [6] experiments. Further investigations of the baryon spectrum are planned or realized, e.g., at J-PARC [7], BGO-OD at ELSA [8], and at the 12 GeV upgrade at JLab allowing studies of the electroproduction of baryon resonances to large four-momentum transfer [9,10].

Partial-wave analysis provides the link between large-scale experimental programs and theory approaches that focus on the intermediate-energy region, where quark confinement manifests itself in a rich spectrum of resonances. Improved and extended techniques are necessary to further our understanding of baryon structure and, in particular, to help

resolve the missing-resonance problem [11]. More generally, the baryon resonance spectrum is tightly related to the open issue of structure formation in QCD, which is arguably the least understood part of the so successful standard model of particle physics. Furthermore, partial-wave analysis provides the bridge to compare experiment with theories and models such as quark models [12–23], Dyson-Schwinger and related approaches [24–39], Roy-Steiner equations [40], chiral perturbation theory (ChPT) with Δ resonance [41–44] or perturbative calculations using the complex-mass scheme [45,46], and chiral unitary calculations [47–62]. For example, in Refs. [47–51] a gauge-invariant implementation of the full Bethe-Salpeter equation was used to fit and predict light S -wave baryons. Specifically for pion electroproduction, ChPT has been successfully applied in the analysis of the threshold region [63–66]. Furthermore, the spectrum of excited baryons has become accessible in lattice QCD calculations [67–78]. The use of meson and baryon-type operators has also enabled the calculation of baryonic scattering amplitudes [79–83] using Lüscher's method [84], see also Ref. [85]. However, so far, little is known about the decay properties of baryons from such first-principles calculations.

While most of the early progress [86–90] in baryon spectroscopy was based on the analysis of meson-nucleon scattering data, particularly pion-nucleon scattering ($\pi N \rightarrow \pi N, \pi N \rightarrow \pi \pi N$), photon-nucleon interactions offer the possibility of detecting unstable intermediate states with small branchings to the πN channel [91]. Many groups have performed either single-channel or multichannel analyses of these photo-induced reactions. In the more recent single-channel analyses, fits have typically used isobar models [92–95] with unitarity constraints at lower energies, K -

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matrix-based formalisms, having built-in cuts associated with opening inelastic channels [96], and dispersion-relation constraints [94,97]. Multichannel fits have analyzed data (or, in some cases, amplitudes) from hadronic scattering data together with the photon-induced channels [98]. These approaches have utilized unitarity more directly. The most active programs are being carried out by the Bonn-Gatchina [99], Jülich-Bonn JüBo [100], ANL-Osaka [101], Kent State [90], JPAC [102], and Gießen [103] groups. At low energies, the chiral MAID analysis provides a comprehensive description of photo- and electroproduction data [104].

With the refinement of dynamical and phenomenological coupled-channel approaches for the analysis of pseudoscalar-meson photoproduction reactions, many new states and their properties could be discovered [1]. In this context, one should emphasize that the only model-independent definition of a resonance is by its properties in the complex energy plane. While the Q^2 variation of resonance couplings is expected to provide a link between perturbative QCD and the region where quark confinement sets in, so far, no unified coupled-channel analysis of photo- and electroproduction experiments exists that simultaneously studies the πN , ηN , and ΔK final states. This study provides a first step in this direction in the form of an analysis of pion electroproduction data.

Going from photo- to electroproduction of pseudoscalar meson, the number of helicity amplitudes increases from four to six, requiring more measurements for the analogous “complete experiment” [105,106], with a multipole decomposition adding longitudinal components to the transverse elements anchored by photoproduction analyses at $Q^2 = 0$. Variations of resonance couplings with Q^2 are expected to provide links between perturbative QCD and regions where quark confinement requires the use of lattice QCD, ChPT, or more phenomenological approaches. Exactly where this transition occurs is not precisely known. The well-known prediction [107] of an $E2/M1$ ratio, for the $\Delta(1232)$ state, approaching unity shows no sign of occurring, remaining essentially flat at a small negative value. See also Ref. [108] for a review. In contrast, other clear resonances, such as the $N(1520)$, do show rapid behavior in the low- Q^2 region, followed by a smooth transition to higher values of Q^2 . The reliable determination of helicity amplitudes for $Q^2 > 0$ is also relevant for neutrino physics [109,110]. See Ref. [111] for recent progress in this direction by the ANL/Osaka group.

Electroproduction experiments, e.g., by the CLAS Collaboration at JLab, are producing a wealth of data that, in many cases, still await a detailed analysis in pion electroproduction [112–115] (JLab), [116] (A1 Collaboration at MAMI), η electroproduction [117] (JLab), [118] (A1 Collaboration at MAMI), and kaon electroproduction [3] (A2 Collaboration), [119] (JLab). It should also be stressed that pion and kaon electroproduction experiments with the new CLAS12 detector at the 12 GeV upgrade of Jefferson Lab [10,120] will provide many data that require a timely analysis.

The ANL-Osaka group extended its dynamical coupled-channel analysis of pion electroproduction [121] to higher Q^2 values [122]. Plots of the $\Delta(1232)$ amplitudes at the resonance pole position (yielding a complex amplitude) also seem to qualitatively reproduce results found for the MAID

and SAID analyses [123]. However, results have generally been restricted to the low-energy $\Delta(1232)$ and $N(1440)$ states.

The most widely used single-pion electroproduction analyses, covering the resonance region, have been performed by the Mainz (MAID) [92,124–127] and Jefferson Lab [2,128–130] groups. An extensive single-pion electroproduction database and a K -matrix-based SAID fit is also available [131]. Eta electroproduction has been analyzed in the Eta MAID framework [95], and kaon electroproduction by the Ghent group [132]. These fits have generally utilized a Regge [133] or Regge-plus-resonance approach [134] at high to medium energies. (We mention here parenthetically that the Ghent Regge approach can be improved by correctly implementing the local gauge-invariance constraints [135].) Effective Lagrangian and isobar models have also been used [136,137], with some of these available via the MAID website [138] for both kaon and eta electroproduction [95].

These are all single-channel analyses with approaches similar to the associated real-photon fits, but generally with the exception of the MAID and SAID analyses, not including the real-photon data as a constraint at $Q^2 = 0$. Both the MAID and Jefferson Lab groups have made fits using Breit-Wigner (BW) plus background models with resonance couplings extended to include a Q^2 dependence. We note that not all resonances can be well described by the BW form, especially if a resonance is located very close to a multiparticle threshold. In the Jefferson Lab analyses [2], a further fit was again based on satisfying fixed- t dispersion relation. It should be mentioned that two-pion electroproduction fits have also been performed and compared with the single-pion results, at Jefferson Lab [139–143]. See Ref. [140] for a review. Remarkably, a new baryon resonance has been claimed in an analysis of combined $\pi\pi N$ photo and electroproduction [144].

In this study we perform a first step towards a truly coupled-channel analysis of electroproduction data. For this, we analyze pion electroproduction data off proton targets in both charge channels for energies $1.13 \text{ GeV} < W < 1.6 \text{ GeV}$. In this work, we do not analyze the threshold region. Mass differences within pion and nucleon multiplets, and necessary checks with ChPT [63–66] require a modified parametrization that has been developed in photoproduction [145] but will be included for electroproduction at a later stage. The upper limit in photon virtuality for data included in the fits varies from $Q^2 = 4 \text{ GeV}^2$ to 6 GeV^2 to assess the stability of the solutions.

Special emphasis is put on gauge invariance and Siegert’s condition [146,147], which is manifestly included in the parametrization. The electroproduction amplitude is constructed such that, at the photon point $Q^2 = 0 \text{ GeV}^2$, it describes pion, η , and $K\Lambda$ photoproduction data in the form of the most recent solution of the Jülich-Bonn analysis effort, the “JüBo2017” solution [148]. The hadronic part of that amplitude describes also various pion-induced reactions. Extensions including η and kaon electroproduction data, as well as the simultaneous fit of photo- and electroproduction data, are left to future research. In this context it will be relevant to revise kaon polarization observables due to recent updates of the fundamental Λ decay parameter α_- [149,150].

This study is organized as follows. Section II contains formal aspects of electroproduction (kinematics, Siegert’s

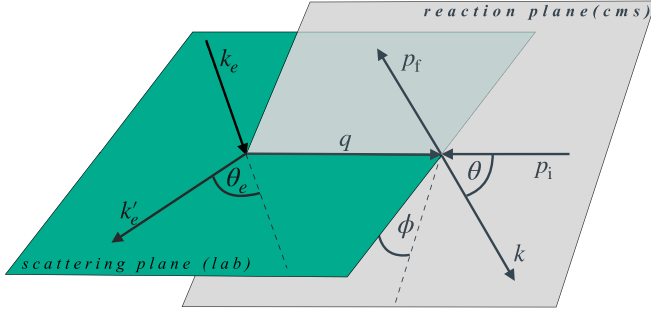


FIG. 1. Kinematics of an electroproduction experiment. The scattering plane is defined by the respective ingoing (outgoing) electron momenta k_e (k'_e) with the electron-scattering angle θ_e . The reaction plane is spanned by the virtual photon and the outgoing meson scattered by an angle θ .

condition, observables, and multipoles), while the parametrization of electroproduction multipoles as an extension of the Jülich-Bonn amplitude is discussed in Sec. III. Results are presented and discussed in Sec. IV, and the conclusions can be found in Sec. V.

II. PRELIMINARIES

A. Kinematics

The pion electroproduction process in question occurs via the following reaction (bold symbols denote three-momenta throughout the paper)

$$\gamma^*(q) + p(p_i) \rightarrow \pi(k) + N(p_f),$$

with the virtual photon $\gamma^*(q)$ being produced via $e_{\text{in}}(k_e) \rightarrow e_{\text{out}}(k'_e) + \gamma^*(q)$. Thus, the momentum transfer $Q^2 = -\omega^2 + q^2$ is non-negative for spacelike processes and acts as an independent kinematical variable in addition to the total energy in the center-of-mass frame W . In this frame, the magnitude of the three-momentum of the photon ($q = |q|$) and produced pion ($k = |k|$) read

$$q = \frac{\sqrt{\lambda(W^2, m^2, -Q^2)}}{2W}, \quad k = \frac{\sqrt{\lambda(W^2, m^2, M^2)}}{2W}, \quad (2.1)$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ denotes the usual Källén triangle function. The pion and nucleon masses are denoted throughout this paper by M and m , respectively. From this, the photon energy is $\omega = (W^2 - m^2 - Q^2)/(2W)$. The angular structure of the above process is depicted in Fig. 1, where θ_e is the angle of the ingoing or outgoing electron in the scattering plane, ϕ is the angle of the reaction plane with respect to the scattering plane, and θ is the center-of-mass scattering angle in the latter plane. The world data on electroproduction is represented with respect to the five variables

$$\mathcal{O}(Q^2, W, \phi, \theta, \epsilon),$$

where ϵ is the transverse photon polarization,

$$\epsilon = 1 + 2 \frac{q_L^2}{Q^2} \tan^2 \frac{\theta_e}{2}, \quad (2.2)$$

where q_L is the photon three-momentum in the laboratory frame. The resulting observables $\mathcal{O}$ will be discussed in the next section.

There are several important limits and thresholds of this kinematics which will be discussed later and thus require an introduction.

- (1) *Photon point.* This corresponds to $Q^2 \equiv 0$, reducing the process to pion photoproduction. In this limiting case, denoted by a subscript γ , the amplitudes are independent of the angle ϕ and

$$\omega_\gamma = q_\gamma = \frac{W^2 - m^2}{2W}. \quad (2.3)$$

The electroproduction amplitudes are constrained by photoproduction data via the multipoles of the JüBo model, see Sec. III A.

- (2) *Production threshold.* This refers to the lowest physical energy of the final meson-baryon pair, i.e., $W = (m + M) \Leftrightarrow k \equiv 0$.
- (3) *Pseudo-threshold.* This denotes an unphysical point $q = 0$ or correspondingly

$$Q_{\text{PT}\pm}^2 = -(W \pm m)^2. \quad (2.4)$$

The particular importance of this kinematic point arises from Siegert's condition, which will serve as a boundary condition for our parametrization of the multipoles, see Sec. III.

B. Transition amplitudes: Multipoles and Siegert's condition

For a general photon-induced photo- or electroproduction of a meson off a nucleon, the transition amplitude in the standard one-photon approximation reads

$$T_{\text{fi}} = \langle f | \left(-i \int d^4r A^\mu J_\mu \right) | i \rangle, \quad (2.5)$$

with J_μ denoting the electromagnetic current, see, e.g., the seminal papers [151–153] for more details. The components of the vector potential A^μ are solutions of the Laplace equation. For instance, for the time component (scalar potential),

$$(\nabla^2 + q^2)A^0 = 0. \quad (2.6)$$

Then, A^0 can be decomposed into contributions with given angular-momentum values,

$$A^0 = \sum_{j_\gamma m_\gamma} \int \frac{dq}{2\pi} \frac{q^2}{\sqrt{\omega}} (a_{j_\gamma m_\gamma} u_{j_\gamma m_\gamma}^C + a_{j_\gamma m_\gamma}^\dagger u_{j_\gamma m_\gamma}^{C*}), \quad (2.7)$$

where j_γ is the angular momentum of the photon and $a_{j_\gamma m_\gamma} \equiv a_{j_\gamma m_\gamma}(q)$ [$a_{j_\gamma m_\gamma}^\dagger \equiv a_{j_\gamma m_\gamma}^\dagger(q)$] are annihilation [creation] operators. Furthermore,

$$u_{j_\gamma m_\gamma}^C \equiv u_{j_\gamma m_\gamma}^C(q, r) = j_{j_\gamma}(qr) Y_{j_\gamma m_\gamma}(\theta, \phi). \quad (2.8)$$

Everywhere, $j_{j_\gamma}(qr)$ and $Y_{lm}(\theta, \phi)$ denote the spherical Bessel functions and spherical harmonics, respectively.

Similarly, the three-vector potential can be expanded in elementary vectors $\mathbf{u}^x$ with a given angular momentum as in

the scalar case,

$$A = \sum_{j_\gamma m_\gamma} \sum_x \int \frac{dq}{2\pi} \frac{q^2}{\sqrt{\omega}} [a_{j_\gamma m_\gamma}^x(q) u_{j_\gamma m_\gamma}^x(\mathbf{q}, \mathbf{r}) + a_{j_\gamma m_\gamma}^{x\dagger}(q) u_{j_\gamma m_\gamma}^{x*}(\mathbf{q}, \mathbf{r})], \quad (2.9)$$

where ω is the photon energy, while $x \in \{E, M, L\}$ labels electric, magnetic, and longitudinal components, respectively. The vectors $\mathbf{u}^x$ can be constructed from the scalar u^C using differential operators,

$$\begin{aligned} u_{j_\gamma m_\gamma}^L(\mathbf{q}, \mathbf{r}) &= \frac{i}{q} \nabla u_{j_\gamma m_\gamma}^C(\mathbf{q}, \mathbf{r}), \\ u_{j_\gamma m_\gamma}^M(\mathbf{q}, \mathbf{r}) &= \frac{\mathbf{r} \times \nabla}{\sqrt{j_\gamma(j_\gamma - 1)}} u_{j_\gamma m_\gamma}^C(\mathbf{q}, \mathbf{r}), \\ u_{j_\gamma m_\gamma}^E(\mathbf{q}, \mathbf{r}) &= \frac{\nabla \times \mathbf{L}}{q\sqrt{j_\gamma(j_\gamma + 1)}} u_{j_\gamma m_\gamma}^C(\mathbf{q}, \mathbf{r}). \end{aligned} \quad (2.10)$$

Because ∇ , $\mathbf{L} = -i\mathbf{r} \times \nabla$, and $\nabla \times \mathbf{L}$ commute with the Laplacian, u^L , u^M , and u^E are also solutions of the Laplace equation and are orthogonal to each other. The corresponding Coulomb (C), longitudinal (L), magnetic (M), and electric (E) multipoles then read for a given photon angular-momentum state

$$\begin{aligned} C_{j_\gamma m_\gamma}(\mathbf{q}) &= \langle f | \int d^3r u_{j_\gamma m_\gamma}^C(\mathbf{q}, \mathbf{r}) \rho(\mathbf{r}) | i \rangle, \\ L_{j_\gamma m_\gamma}(\mathbf{q}) &= \langle f | \int d^3r u_{j_\gamma m_\gamma}^L(\mathbf{q}, \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) | i \rangle, \\ E_{j_\gamma m_\gamma}(\mathbf{q}) &= \langle f | \int d^3r u_{j_\gamma m_\gamma}^E(\mathbf{q}, \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) | i \rangle, \\ M_{j_\gamma m_\gamma}(\mathbf{q}) &= \langle f | \int d^3r u_{j_\gamma m_\gamma}^M(\mathbf{q}, \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) | i \rangle, \end{aligned} \quad (2.11)$$

respectively. Only three multipoles here are independent because of the continuity equation for the current, thus relating scalar and longitudinal multipoles according to

$$\omega C_{j_\gamma}(\mathbf{q}) = q L_{j_\gamma}(\mathbf{q}), \quad (2.12)$$

where the m_γ dependence of the multipoles cancels out by the Wigner-Eckart theorem, see, e.g., Ref. [153].

Finally, and following Ref. [153], we note that in the long-wavelength limit ($q \rightarrow 0$) (pseudothreshold) $j_{j_\gamma}(qr) \rightarrow (qr)^{j_\gamma}/(2j_\gamma + 1)!!$. Using then Eq. (2.8) and $\nabla \times \mathbf{L}(r^{j_\gamma} Y_{j_\gamma m_\gamma}) = i(j_\gamma + 1) \nabla(r^{j_\gamma} Y_{j_\gamma m_\gamma})$ yields straightforwardly

$$u_{j_\gamma}^E(\mathbf{k}, \mathbf{r}) \rightarrow \frac{i}{q} \sqrt{\frac{j_\gamma + 1}{j_\gamma}} \nabla u_{j_\gamma}^C(\mathbf{q}, \mathbf{r}). \quad (2.13)$$

Then, substituting this into Eqs. (2.10) and (2.11) results in an exact relation between electric and longitudinal multipole at the pseudothreshold,

$$E_{j_\gamma} = \sqrt{\frac{j_\gamma + 1}{j_\gamma}} L_{j_\gamma}, \quad \text{at } q = 0, \quad (2.14)$$

called Siegert's theorem and that provides an important constraint on our parametrization of the electroproduction

multipoles. The practical implementation of it will be discussed in the next section.

C. Transition amplitudes: Chew, Goldberger, Low, and Nambu (CGLN) and helicity amplitudes

To find a practical access to the (ELM) multipoles (2.11), introduced above, we follow the seminal works [151,154] utilizing the nomenclature of Ref. [152]. Taking the z axis as the quantization axis and working in the center of mass of the final pion-nucleon system yields the general Lorentz covariant transition matrix element

$$T_{fi} = 8\pi W \chi_f^\dagger \sum_{a=1}^6 (\mathcal{F}_a \mathcal{G}_a) \chi_i. \quad (2.15)$$

Note that charge conservation is already implemented here thus reducing the number of independent structures to six. Here, χ denotes the two-component spinor, and $\mathcal{F}_a$ are the CGLN amplitudes,¹ being coefficients of

$$\begin{aligned} \mathcal{G} &= \{i(\boldsymbol{\sigma} \cdot \mathbf{a}), (\boldsymbol{\sigma} \cdot \hat{\mathbf{k}})(\boldsymbol{\sigma} \cdot (\hat{\mathbf{q}} \times \mathbf{a})), \\ &\quad \times i(\boldsymbol{\sigma} \cdot \hat{\mathbf{q}})(\hat{\mathbf{k}} \cdot \mathbf{a}), i(\boldsymbol{\sigma} \cdot \hat{\mathbf{k}})(\hat{\mathbf{k}} \cdot \mathbf{a}), \\ &\quad \times i(\boldsymbol{\sigma} \cdot \hat{\mathbf{q}})(\hat{\mathbf{q}} \cdot \mathbf{a}), i(\boldsymbol{\sigma} \cdot \hat{\mathbf{k}})(\hat{\mathbf{q}} \cdot \mathbf{a})\} \end{aligned} \quad (2.16)$$

with $a_\mu = \epsilon_\mu - (\epsilon_0/\omega)q_\mu$ (ϵ_0 being the zero-component of the polarization vector ϵ_μ [151]), and hat denoting the normalization of the respective three-vector.

The six types of transitions (2.15) can be related to the eigenamplitudes of definite parity and relative angular momentum of the pion-nucleon pair (ℓ) [155]. These amplitudes are identified with electric $E_{\ell\pm}$, magnetic $M_{\ell\pm}$, and scalar or Coulomb multipoles (2.12), see Eqs. (2.11). Expanding with respect to the Legendre polynomials $P_\ell(\cos \theta)$ yields

$$\begin{aligned} \mathcal{F}_1 &= \sum_{\ell \geq 0} \{(\ell M_{\ell+} + E_{\ell+})P'_{\ell+1} \\ &\quad + [(\ell + 1)M_{\ell-} + E_{\ell-}]P'_{\ell-1}\}, \\ \mathcal{F}_2 &= \sum_{\ell \geq 1} [(\ell + 1)M_{\ell+} + \ell M_{\ell-}]P'_\ell, \\ \mathcal{F}_3 &= \sum_{\ell \geq 1} [(E_{\ell+} - M_{\ell+})P''_{\ell+1} + (E_{\ell-} - M_{\ell-})P''_{\ell-1}], \\ \mathcal{F}_4 &= \sum_{\ell \geq 2} (M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-})P''_\ell, \\ \mathcal{F}_5 &= \sum_{\ell \geq 0} [(\ell + 1)L_{\ell+}P'_{\ell+1} - \ell L_{\ell-}P'_{\ell-1}], \\ \mathcal{F}_6 &= \sum_{\ell \geq 1} [\ell L_{\ell-} - (\ell + 1)L_{\ell+}]P'_\ell. \end{aligned} \quad (2.17)$$

Here we have suppressed the dependence of the multipoles on W and Q^2 for simplicity. The total angular momentum

¹Referring to the authors of Ref. [154] (Chew, Goldberger, Low, Nambu), those amplitudes were originally derived for the photoproduction amplitudes but extended later by Dennery [151] to electroproduction.

is given by $J = \ell \pm 1/2 = \ell \pm$. The multipole decomposition yields a certain behavior of the multipoles at the physical and pseudothreshold,

			$k \rightarrow 0$	$q \rightarrow 0$	
$E_{\ell+}$	and	$L_{\ell+}$	for $\ell \geq 0$	k^ℓ	q^ℓ
$L_{\ell-}$			for $\ell = 1$	k	q
$M_{\ell+}$	and	$M_{\ell-}$	for $\ell \geq 1$	k^ℓ	q^ℓ
$E_{\ell-}$	and	$L_{\ell-}$	for $\ell \geq 2$	k^ℓ	$q^{\ell-2}$

(2.18)

Finally, coming back to Siegert's theorem (2.14), we note that the multipoles in the previous section are labeled with the incident-photon angular momentum j_γ , while in the current section they are indexed by the final-state orbital angular momentum ℓ . Transforming to the latter basis Siegert's theorem takes the form [147]

$$\frac{E_{\ell+}}{L_{\ell+}} = 1 \text{ and } \frac{E_{\ell-}}{L_{\ell-}} = \frac{\ell}{1-\ell}, \text{ at } q = 0, \quad (2.19)$$

which is also the form employed in this work at $Q^2 = Q_{\text{PT}-}^2$, referring to it as Siegert's condition.² This is crucial, since the current parametrization relies on the continuation of the available photoproduction solution to $Q^2 > 0$, see Sec. III. The latter, however, do not restrict the longitudinal multipoles. Equating the latter to the electric multipole at the pseudothreshold provides a solution to this problem and is employed in this work.

D. Response functions and observables

Free parameters for the various multipoles will be determined by fits to data of differential cross sections and other observables. These observables are written in terms of response functions, $R(W, Q^2, \theta)$ which can be related to the transition amplitudes, conveniently employing the helicity formalism [156]. In particular, the differential cross section is a function of five kinematic variables discussed in Sec. II A ($W, Q^2, \theta, \phi, \epsilon$) defined as

$$\frac{d\sigma}{d\Omega_f dE_f d\Omega} = \left(\frac{\alpha}{2\pi^2} \frac{E_f}{E_i} \frac{q_L}{Q^2} \frac{1}{1-\epsilon} \right) \frac{d\sigma^v}{d\Omega}, \quad (2.20)$$

where Ω refers to the angles of the final meson baryon system (θ, ϕ) and Ω_f are the angles of the final electron at energy E_f . The differential cross section $d\sigma^v/d\Omega$ for the virtual photon subprocess is commonly further decomposed as

$$\begin{aligned} \frac{d\sigma^v}{d\Omega} = & \sigma_T + \epsilon \sigma_L + \sqrt{2\epsilon(1+\epsilon)} \sigma_{LT} \cos \phi \\ & + \epsilon \sigma_{TT} \cos 2\phi + h \sqrt{2\epsilon(1-\epsilon)} \sigma_{LT'} \sin \phi, \end{aligned} \quad (2.21)$$

where h is the helicity of the incoming electron. The quantities $\{\sigma_x | x = (T, TT, LT, L, LT')\}$ are referred to as structure functions. Data involving polarized quantities are included from Jefferson Lab experiments: (1) From Refs. [157,158] via longitudinal-transverse structure functions $\sigma_{LT'}$ and (2) the

$K_{1D} = \{K_{1D}^X | X = A, B, \dots, T\}$ observables from Ref. [159] related to the response functions as shown in Table II in Appendix 2. In general, and following the convention of Ref. [105], the structure functions in Eq. (2.21) are obtained from the response functions R . The latter denote the coefficients that expand the azimuthal angle dependence of the general differential cross section of an electroproduction reaction when all polarizations are taken into account, see, e.g., Ref. [105]. In our case, and using $q_\gamma \equiv q(Q^2 = 0)$ from Eq. (2.3), the required connection reduces to

$$\begin{aligned} \sigma_T &= \frac{k}{q_\gamma} R_T^{00}, \quad \sigma_L = \frac{k}{q_\gamma} \frac{Q^2}{\omega^2} R_L^{00}, \quad \sigma_{TT} = \frac{k}{q_\gamma} R_{TT}^{00}, \\ \sigma_{LT} &= \frac{k}{q_\gamma} \frac{\sqrt{Q^2}}{\omega} R_{LT}^{00}, \quad \sigma_{LT'} = \frac{k}{q_\gamma} \frac{\sqrt{Q^2}}{\omega} R_{LT'}^{00}. \end{aligned} \quad (2.22)$$

The response functions can be expressed in terms of helicity amplitudes (H),

$$\begin{aligned} R_T^{00} &= (|H_1|^2 + |H_2|^2 + |H_3|^2 + |H_4|^2)/2, \\ R_L^{00} &= (|H_5|^2 + |H_6|^2), \\ R_{LT}^{00} &= [(H_1 - H_4)H_5^* + (H_2 + H_3)H_6^*]/\sqrt{2}, \\ R_{TT}^{00} &= \text{Re}(H_3H_2^* - H_4H_1^*), \\ R_{LT'}^{00} &= \text{Im}[(H_4 - H_1)H_5^* - (H_2 + H_3)H_6^*]/\sqrt{2}, \\ R_{LT}^{0Y} &= -\text{Re}[(H_2 + H_3)H_5^* + (H_4 - H_1)H_6^*]/\sqrt{2}, \end{aligned} \quad (2.23)$$

where where the connection to CGLN $\mathcal{F}$ amplitudes is given by

$$\begin{aligned} H_1 &= \sin \theta \cos \frac{\theta}{2} (-\mathcal{F}_3 - \mathcal{F}_4)/\sqrt{2}, \\ H_2 &= \sqrt{2} \cos \frac{\theta}{2} \left[\mathcal{F}_2 - \mathcal{F}_1 + (\mathcal{F}_3 - \mathcal{F}_4) \sin^2 \frac{\theta}{2} \right], \\ H_3 &= \sin \theta \sin \frac{\theta}{2} (\mathcal{F}_3 - \mathcal{F}_4)/\sqrt{2}, \\ H_4 &= \sqrt{2} \sin \frac{\theta}{2} \left[\mathcal{F}_1 + \mathcal{F}_2 + (\mathcal{F}_3 + \mathcal{F}_4) \cos^2 \frac{\theta}{2} \right], \\ H_5 &= \cos \frac{\theta}{2} (\mathcal{F}_5 + \mathcal{F}_6), \\ H_6 &= \sin \frac{\theta}{2} (\mathcal{F}_6 - \mathcal{F}_5), \end{aligned} \quad (2.24)$$

following again the phase convention of Refs. [156,160]. Several other observables are included in the fits, as described in Sec. IV A:

$$\begin{aligned} P_Y &= -\sqrt{2\epsilon(1+\epsilon)} \frac{\omega}{\sqrt{Q^2}} \frac{R_{LT}^{0Y}}{R_T^{00} + \epsilon\omega^2/Q^2 R_L^{00}}, \\ \rho_{LT} &= \sqrt{2\epsilon(1+\epsilon)} \frac{R_{LT}^{00}}{R_T^{00} + \epsilon(R_L^{00} + R_{TT}^{00})}, \\ \rho_{LT'} &= \sqrt{2\epsilon(1-\epsilon)} \sin \phi \frac{\sigma_{LT'}}{d\sigma^v/d\Omega}, \end{aligned} \quad (2.25)$$

while the K_{D1} observables are given explicitly in Appendix 2.

²We do not employ the same condition at the second pseudothreshold, since it is located much further away from the physical region.

III. MULTIPOLE PARAMETRIZATION

A. The Jülich-Bonn dynamical coupled-channel approach

The input at the photon point is provided by the Jülich-Bonn (JüBo) framework, a dynamical coupled-channel approach that aims at the extraction of the nucleon resonance spectrum in a combined analysis of pion- and photon-induced hadronic reactions. In this approach, two-body unitarity and analyticity are respected and the baryon resonance spectrum is determined in terms of poles in the complex energy plane on the second Riemann sheet. A detailed description of the model can be found in Refs. [145,161] and references therein.

The purely hadronic scattering process of a meson-baryon pair ν is described in a field-theoretical framework by potentials $V_{\mu\nu}$, which are derived from a chiral Lagrangian and iterated in a Lippmann-Schwinger equation

$$T_{\mu\nu}(k, p', W) = V_{\mu\nu}(k, p', W) + \sum_{\kappa} \int_0^{\infty} dp p^2 V_{\mu\kappa}(k, p, W) \times G_{\kappa}(p, W) T_{\kappa\nu}(p, p', W), \quad (3.1)$$

where the indices μ , ν , and κ denote the outgoing, incoming, and intermediate meson-baryon channels, respectively. The model incorporates the two-body channels πN , ηN , $K\Lambda$, and $K\Sigma$ and the channels ρN , σN , and $\pi\Delta$, which effectively parametrize the $\pi\pi N$ channel. In Eq. (3.1), k (p') indicates the modulus of the outgoing (incoming) three-momentum in the center-of-mass system, which can be on- or off-shell. The propagator G_{κ} is given by

$$G_{\kappa}(p, W) = \frac{1}{W - E_a(p) - E_b(p) + i\epsilon}, \quad (3.2)$$

with the on-mass-shell energies $E_a = (m_a^2 + p^2)^{1/2}$ and $E_b = (m_b^2 + p^2)^{1/2}$ of the intermediate particles a and b in channel κ with masses m_a and m_b . While Eq. (3.2) applies to the channels $\kappa = \pi N$, ηN , $K\Lambda$, or $K\Sigma$, the propagator is of a more complex form for channels with unstable particles, i.e., ρN , σN , and $\pi\Delta$ [162,163]. The scattering potential $V_{\mu\nu}$ is constructed from s -channel processes that account for genuine resonances, t - and u -channel exchanges of mesons and baryons, and contact diagrams that are included to absorb physics beyond the explicitly included processes.

The photoproduction process is described in the semiphenomenological approach of Ref. [145], where the electric or magnetic photoproduction multipole amplitude [see Eq. (2.11)] is given by

$$\mathcal{M}_{\mu\gamma}(k, W) = V_{\mu\gamma}(k, W) + \sum_{\kappa} \int_0^{\infty} dp p^2 T_{\mu\kappa}(k, p, W) \times G_{\kappa}(p, W) V_{\kappa\gamma}(p, W). \quad (3.3)$$

Here, the index γ denotes the initial γN channel and μ (κ) the final (intermediate) meson-baryon pair, while $T_{\mu\kappa}$ is the hadronic half-off-shell matrix of Eq. (3.1) and k denotes, again, the momentum of the outgoing meson.

The photoproduction kernel $V_{\mu\gamma}$ is constructed as

$$V_{\mu\gamma}(p, W) = \alpha_{\mu\gamma}^{\text{NP}}(p, W) + \sum_i \frac{\gamma_{\mu,i}^a(p) \gamma_{\gamma,i}^c(W)}{W - m_i^b}, \quad (3.4)$$

where $\gamma_{\gamma,i}^c$ describes the interaction of the photon with the resonance state i with bare mass m_i^b and $\alpha_{\mu\gamma}^{\text{NP}}$ accounts for the coupling of the photon to the so-called background or non-pole part of the amplitude. Both quantities are parametrized by energy-dependent polynomials. See Appendix 3 for details. In particular, we note that the hadronic resonance vertex function $\gamma_{\mu,i}^a$ in Eq. (3.4) is the same as in the hadronic scattering potential to ensure the cancellation of the poles in Eq. (3.4). Explicit expressions for $\gamma_{\mu,i}^a$ can be found in Ref. [164].

The hadronic scattering potential $V_{\mu\nu}$ and the polynomials in $\gamma_{\gamma,i}^c$ and $\alpha_{\mu\gamma}^{\text{NP}}$ contain free parameters that are fit to the data in a χ^2 minimization using MINUIT on the JURECA supercomputer at the Jülich Supercomputing Centre [165]. In its most recent form [148] the JüBo model describes the reactions $\pi N \rightarrow \pi N$, ηN , $K\Lambda$, and $K\Sigma$ in addition to pion, eta, and $K^+\Lambda$ photoproduction off the proton. More than 48 000 data points were analyzed in simultaneous fits and the N and Δ spectrum was determined.

B. Extension of the JüBo formalism to electroproduction

To include electroproduction reactions in the JüBo formalism, the photoproduction formalism outlined in the previous section is extended to handle virtual photons with $Q^2 > 0$. Following Eq. (3.3) we first introduce a generic function ($\bar{\mathcal{M}}$) for each electromagnetic multipole ($\mathcal{M}_{\mu\gamma^*} \in \{E_{\mu}, L_{\mu}, M_{\mu}\}$) as

$$\begin{aligned} \bar{\mathcal{M}}_{\mu\gamma^*}(k, W, Q^2) \\ = V_{\mu\gamma^*}(k, W, Q^2) + \sum_{\kappa} \int_0^{\infty} dp p^2 T_{\mu\kappa}(k, p, W) \\ \times G_{\kappa}(p, W) V_{\kappa\gamma^*}(p, W, Q^2), \end{aligned} \quad (3.5)$$

with $\kappa \in \{\pi N, \eta N, K\Lambda, K\Sigma, \pi\Delta, \rho N\}$ and γ^* denoting the ingoing $\gamma^* N$ state. The electroproduction kernel $V_{\mu\gamma^*}$ in Eq. (3.5) is parametrized as

$$\begin{aligned} V_{\mu\gamma^*}(p, W, Q^2) = \alpha_{\mu\gamma^*}^{\text{NP}}(p, W, Q^2) \\ + \sum_{i=1}^{i_{\text{max}}} \frac{\gamma_{\mu,i}^a(p) \gamma_{\gamma^*,i}^c(W, Q^2)}{W - m_i^b}, \end{aligned} \quad (3.6)$$

introducing the Q^2 dependence via a separable ansatz,

$$\begin{aligned} \alpha_{\mu\gamma^*}^{\text{NP}}(p, W, Q^2) = \tilde{F}^{\mu}(Q^2) \alpha_{\mu\gamma}^{\text{NP}}(p, W), \\ \gamma_{\gamma^*,i}^c(W, Q^2) = \tilde{F}_i(Q^2) \gamma_{\gamma,i}^c(W), \end{aligned} \quad (3.7)$$

with a channel-dependent form factor $\tilde{F}^{\mu}(Q^2)$ and another channel-independent form factor $\tilde{F}_i(Q^2)$ that depends on the resonance number i . Note that the channel dependence is inherited from the structure of the photoproduction ansatz of Eq. (3.4), which separates the photon-induced vertex (γ^c) from the decay vertex of a resonance to the final meson-baryon pair (γ_{μ}^a).

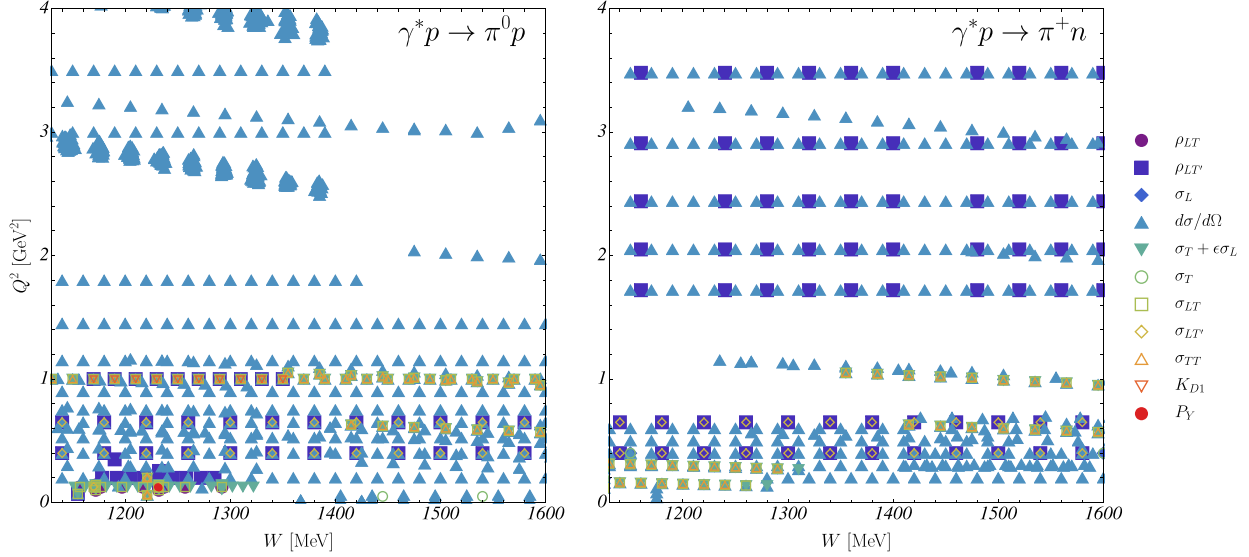


FIG. 2. Overview of the experimental data [157–159,170–212] used in the main fits ($0 < Q^2 < 4 \text{ GeV}^2$, $1.13 < W < 1.6 \text{ GeV}$) for aggregated values of θ , ϕ , ϵ . Symbol shapes differentiate between different observable types, while the total number of data is 82 968. See Sec. IID and Appendix 2 for the definition of the observables.

Both $\tilde{F}^\mu(Q^2)$ and $\tilde{F}_i(Q^2)$ are chosen as

$$\tilde{F}(Q^2) = \tilde{F}_D(Q^2) e^{-\beta_0 Q^2/m^2} P^N(Q^2/m^2), \quad (3.8)$$

where

$$\tilde{F}_D(Q^2) = \frac{1}{(1 + Q^2/b^2)^2} \frac{1 + e^{-Q_r^2/Q_w^2}}{1 + e^{-(Q^2 - Q_r^2)/Q_w^2}} \quad (3.9)$$

is a combination of the empirical dipole form factor with $b^2 = 0.71 \text{ GeV}^2$, usually implemented in such problems, see, e.g., Ref. [166], as well as a Woods-Saxon form factor with $Q_w^2 = 0.5 \text{ GeV}^2$ and $Q_r^2 = 4.0 \text{ GeV}^2$, which is introduced to ensure that, at large Q^2 , the multipoles vanish sufficiently rapidly. Furthermore, the polynomial $P^N(x) = 1 + \beta_1 x + \beta_2 x^2 + \dots + \beta_N x^N$ is to be fit to data, along with the parameter β_0 . A similar parametrization is chosen in Refs. [122,167].

For electric and magnetic multipoles, the quantities $\gamma_{\mu,i}^a$, $\alpha_{\mu\gamma}^{NP}$, $\gamma_{\gamma,i}^c$, and m_i^b in the electroproduction amplitude of Eqs. (3.6) and (3.7), as well as $T_{\mu\kappa}$ in Eq. (3.5), represent the input at the photon point in the current analysis. Numerical values are taken from the JüBo2017 solution of Ref. [148].

For longitudinal multipoles there is no information on $\alpha_{\mu\gamma}^{NP}$ or $\gamma_{\gamma,i}^c$ at the photon point. To overcome this we employ the following strategy:

- (1) Following Siegert's condition (2.19), we apply

$$\alpha_{L\pm}^{NP}(Q^2) = \frac{\omega(Q^2)}{\omega(Q_{PT-}^2)} \frac{\alpha_{E\pm}^{NP}(Q_{PT-}^2)}{\tilde{F}_D(Q_{PT-}^2)} \tilde{F}_D(Q^2) D_{\ell\pm}(Q^2), \quad (3.10)$$

and similarly for $\gamma_{\gamma^*,i}$. The photon energy ω was defined below Eq. (2.1). The new functions $D_{\ell\pm}(Q^2)$ incorporate Siegert's condition exactly, ensuring at the

same time a Q^2 falloff behavior. Explicitly they read

$$D_{\ell+}(Q^2) = e^{-\beta_0 q/q_\gamma} P^N(q/q_\gamma),$$

$$D_{\ell-}(Q^2) = -\frac{\ell-1}{\ell} e^{-\beta_0 q/q_\gamma} P^N(q/q_\gamma), \quad (3.11)$$

where $q_\gamma = q|_{Q^2=0}$ from Eq. (2.1).

- (2) For the two cases with vanishing electric multipole, i.e., $(\ell\pm, I) = (1-, 1/2)$ and $(\ell\pm, I) = (1-, 3/2)$, the longitudinal multipole is obtained from the magnetic one via

$$\alpha_{L\pm}^{NP}(Q^2) = \zeta^{NP} \frac{\omega(Q^2)}{\omega(Q_{PT-}^2)} \tilde{F}^\mu(Q^2) \alpha_{M\pm}^{NP}(p, W), \quad (3.12)$$

and similarly for $\gamma_{\gamma^*,i}$. The new real-valued normalization constants ζ^{NP} will be determined from the fit.

Imposing the pseudothreshold constraints of Table (2.18), a q - and ℓ -dependent factor is introduced in the parametrization of $\mathcal{M}_{\mu\gamma^*}$,

$$\mathcal{M}_{\mu\gamma^*}(k, W, Q^2) = R_{\ell'}(\lambda, q/q_\gamma) \tilde{\mathcal{M}}_{\mu\gamma^*}(k, W, Q^2), \quad (3.13)$$

with q_γ of Eq. (2.3); λ is a parameter to be fit, and

$$R_{\ell'}(\lambda, x) = \frac{B_{\ell'}(\lambda x)}{B_{\ell'}(\lambda)}, \quad (3.14)$$

with the Blatt-Weisskopf barrier-penetration factors $B_{\ell'}(r)$ [168,169] with the limits

$$B_{\ell'}(r) \approx O(r^{\ell'}) \text{ and } B_{\ell'}(r) \approx O(r^0) \quad (3.15)$$

for small and large arguments of $B_{\ell'}(r)$, respectively. The index ℓ' relates to $J = \ell \pm$ as

$$\ell' = \begin{cases} \ell & \text{for } (E_{\ell+}, L_{\ell+}, L_{1-}, M_{\ell+}, M_{\ell-}) \\ \ell - 2 & \text{for } (E_{\ell-}, L_{\ell-} \text{ and } \ell \geq 2), \end{cases} \quad (3.16)$$

which ensures that the pseudthreshold constraints tabulated in (2.18) are satisfied. Explicit forms of $B_\ell(r)$ for values of ℓ from zero to five are given in Appendix 1. Note that the transition from small to large arguments in Eq. (3.15) should happen on a natural scale, i.e., λ , which is unitless, should be of order one. The threshold behavior $\mathcal{M} \sim k^\ell$ tabulated in (2.18) is automatically satisfied by electroproduction multipoles because it is already built into the photoproduction amplitude, and this carries over to the virtual-photon case.

To summarize the structure of the fit, each multipole $E_{\mu\gamma^*}$ and $M_{\mu\gamma^*}$ carries $(1+N)$ fit parameters $\beta_0, \dots, \beta_N$ for the non-pole part $\tilde{F}^\mu$, plus $(1+N)$ parameters for each $\tilde{F}_i$ of the $i_{\max}$ resonances in the pertinent partial wave. The channel dependence of the non-pole form factor $\tilde{F}^\mu(Q^2)$ is not fully used in the current fits, because only pion electroproduction data are analyzed. In $\tilde{F}^\mu$ we therefore set those β_i to zero that couple to the photon but do not correspond to the πN channel, i.e., $(\mu \in \{\eta N, K\Lambda, \pi\Delta\} [148])$.

Furthermore, one could also fit the parameters Q_w and Q_r of Eq. (3.9) to data, but we chose to fix them to the quoted values to avoid over-parametrization. In addition, there is one Blatt-Weisskopf range factor λ per multipole. The longitudinal multipoles exhibit the exact same structure of fit parameters as the E and M multipoles, through the functions $D_{\ell\pm}$ of Eq. (3.11). For the two exceptions $[(\ell\pm, I) \in \{(1-, 1/2), (1-, 3/2)\}]$, there are $(1 + i_{\max})$ additional fit parameters ζ^{NP} for each longitudinal multipole, as indicated in Eq. (3.12) for the non-pole part and, similarly, for each of the $i_{\max}$ resonances. In total, we allow for 209 fit parameters in the parametrization, but we have also explored variants, as discussed in Sec. IV B.

IV. FITS TO DATA

A. Database

The data used in this study were taken from the extensive SAID database [213], containing on the order of 10^5 data for the electroproduction of charged and neutral pions off proton targets. See Fig. 2 for an overview of the coverage of the Q^2 - W plane as well as the number of data points for each observable. More measurements, both in number and type, are available for neutral-pion than for charged-pion electroproduction. For both final states, the SAID database is dominated by unpolarized differential cross sections.

Statistically, it is better to fit differential cross sections instead of their components according to Eq. (2.20) because, on the one hand, it avoids potential bias in the extraction of the components, and, on the other hand, the data of the components are necessarily correlated but those correlations are typically not quoted in experimental papers. Therefore, we have generally fit differential cross sections directly, if available, rather than the separated components σ_T , σ_L , σ_{TT} , σ_{LT} , and $\sigma_{LT'}$. However, to have simpler comparisons with previous analyses, we have included plots of predicted structure function data as well, see Sec. IV C.

The largest sets of polarized data included in this fit involve ratios of structure functions (2.25) and the recoil-polarization measurements of Kelly. The Kelly data [159] give substantial

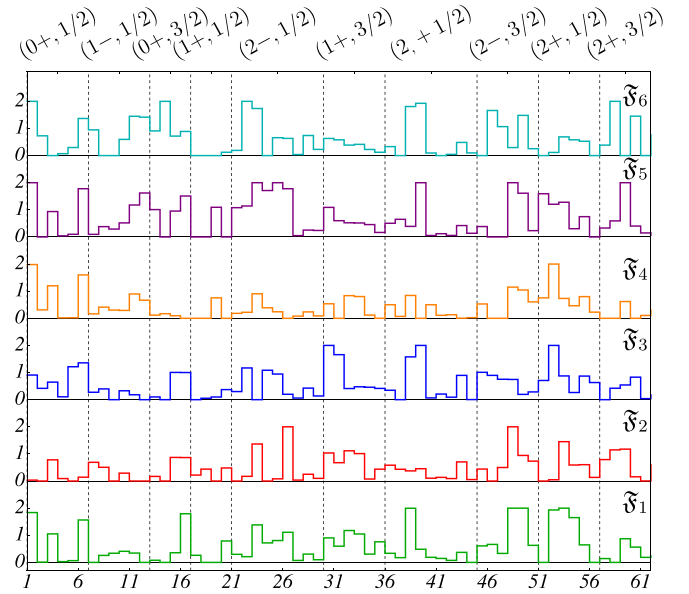


FIG. 3. Values of 62 β_0 parameters with respect to different fit scenarios ($\mathcal{S}_i$) and partial wave $(\ell\pm, I)$, where I denotes isospin.

constraints on neutral-pion production at $Q^2 = 1 \text{ GeV}^2$. In a recent study [105] of the data types required for model-independent complete-experiment or partial-wave analyses, a subset of the Kelly data was shown to be sufficient for a partial-wave analysis up to P waves. It should be noted, however, that this study of experimental completeness assumed error-free measurements. In the Kelly paper [214], fits with different angular-momentum cutoffs were attempted, showing sensitivity to approximations made in their multipole analysis.

Substantial use of SAID and the MAID electroproduction websites [138,213] and codes allowed checks for consistency of conventions, definitions, and naming schemes within the SAID database. As described below, a website is being constructed to compare fits by MAID, SAID, and the present analysis to plotted data [215].

B. Fit scenarios

The parametrization of the multipoles introduced in Sec. III B is subject to a large set of free parameters. In particular, considering S , P , and D waves and both isospin channels $I = (1/2, 3/2)$ leads to 62 new parameters for each new order in the expansion of P^N used to parametrize E , M , and L multipoles. In this context, our preliminary fits have shown that $N = 2$ [see Eqs. (3.8) and (3.11)] yields a sufficiently flexible parametrization, without clear over-fitting of data. See discussion at the end of Sec. III B for additional details. More extensive statistical studies are beyond the scope of the present paper and will be discussed in a future work.

In addition to the β -type parameters, pseudthreshold regulating parameters λ and normalization factors ζ for two longitudinal multipoles ($L_{1-,1/2}$ and $L_{1-,3/2}$) not fixed by Siegert's conditions add 18 and 5 new parameters, respectively. Thus, the total number of parameters sums up to 209.

TABLE I. Fit quality for various fit scenarios described in the main text. All numbers represent χ^2 contributions with respect to different channels and observable types as defined in Eq. (4.1). Total number of data is 82 968 and 91 896 for scenarios 1, 3, 4, 5 and 2, 6, respectively. Cases with no data are marked by “—.” Rightmost column shows the aggregated χ^2_{dof} .

Fit	σ_L		$d\sigma/d\Omega$		$\sigma_T + \epsilon\sigma_L$		σ_T		σ_{LT}		$\sigma_{LT'}$		σ_{TT}		K_{D1}		P_Y		ρ_{LT}		$\rho_{LT'}$		χ^2_{dof}
	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	
$\mathfrak{F}_1$	—	9	65 355	53 229	870	418	87	88	1212	133	862	762	4400	251	4493	—	234	—	525	—	3300	10 294	1.77
$\mathfrak{F}_2$	—	4	69 472	55 889	1081	619	65	78	1780	150	1225	822	4274	237	4518	—	325	—	590	—	3545	10 629	1.69
$\mathfrak{F}_3$	—	8	66 981	54 979	568	388	84	95	1863	181	1201	437	3934	339	4296	—	686	—	687	—	3556	9377	1.81
$\mathfrak{F}_4$	—	22	63 113	52 616	562	378	153	107	1270	146	1198	1015	4385	218	5929	—	699	—	604	—	3548	11 028	1.78
$\mathfrak{F}_5$	—	20	65 724	53 340	536	528	125	81	1507	219	1075	756	4134	230	5236	—	692	—	554	—	3580	11 254	1.81
$\mathfrak{F}_6$	—	18	71 982	58 434	1075	501	29	68	1353	135	1600	1810	3935	291	5364	—	421	—	587	—	3932	11 475	1.78

The free parameters are fit to reproduce the database described in Sec. IV A by minimizing the χ^2 function

$$\chi^2 = \sum_{i=1}^{N_{\text{data}}} \left(\frac{\mathcal{O}_i^{\text{exp}} - \mathcal{O}_i}{\Delta_i^{\text{stat}} + \Delta_i^{\text{syst}}} \right)^2. \quad (4.1)$$

To report the results we also define $\chi^2_{\text{dof}} = \chi^2/(N_{\text{data}} - 209)$. We note that inclusion of systematic errors can be done at different levels of rigor. For example, the SAID group allows data to be “floated” with a χ^2 penalty determined by the overall systematic error [216]. The JüBo/GW group used similar normalization freedom in some of the more recently included data sets [217,218]. In this study, we add the systematic error to the statistical one as indicated in Eq. (4.1) following a procedure widely used in the field; effectively, one neglects the correlations between data due to systematic effects. As there is no reason to believe that systematic effects should be Gaussian, we simply add the uncertainties linearly. In some baryon resonance analyses data are weighted with factors enhancing the influence of sparsely measured observables. In the present study, we did not follow this procedure.

Given the high dimensionality of the minimization problem, an obvious question arises about the statistical significance of the present solution. To our knowledge, a systematic way to answer this question does not exist. Thus, to get an understanding of the χ^2 landscape we have performed a series of fits—denoted by $\mathfrak{F}_i$ with respect to kinematic ranges, and strategy of a minimization. For example, in $\mathfrak{F}_1$ vs $\mathfrak{F}_3$ vs $\mathfrak{F}_4$, we have studied the importance of the choice of starting values by performing preliminary fits to various subsets of data, i.e., a 1/32 random sample, a 1/128 random sample, and the complete set, respectively. In $\{\mathfrak{F}_1, \dots, \mathfrak{F}_4\}$ vs $\{\mathfrak{F}_5, \mathfrak{F}_6\}$ we have changed the fitting strategy from sequential fits (increasing stepwise the number of free parameters from 32 to 209 by including higher and higher partial waves) to simultaneous fits with 209 parameters, all of them set to zero initially. Finally, we have also checked in $\mathfrak{F}_1$ vs $\mathfrak{F}_2$ and in $\mathfrak{F}_5$ vs $\mathfrak{F}_6$ the stability of the results when adding more data by increasing the limit $Q^2 < 4 \text{ GeV} \rightarrow Q^2 < 6 \text{ GeV}$ corresponding to $N_{\text{data}} = 82968 \rightarrow N_{\text{data}} = 91896$. We expect that systematic effects associated with these strategies significantly exceed the statistical uncertainties.

C. Fit results

The quality of the obtained best fits $\mathfrak{F}_i$ is recorded in Table I. There we also record contributions from each observable type to the total χ^2 . We note that the best χ^2 is found to be consistent among all fit strategies, which indicates the overall flexibility of the parametrization. As to the question whether the found minima are identical, the 209-dimensional parameter space is unwieldy to use in addressing this question. Instead, a cut through the 62 dimensions of the crucial β_0 parameters, shown in Fig. 3, suggests no clear similarity between the best-fit parameters of individual fits. While any conclusion drawn from this observation is prone to possibly large correlations between different parameters, it seems that the solutions shown in Table I represent different local minima of the χ^2 landscape. This strengthens our previous assumption that including various fit scenarios is indeed a fair representation of the uncertainties.

The fit results suggest that the approach is sufficiently flexible to provide a good overall fit to the data. Going beyond arguments based on the overall χ^2 , we display the fit quality for selected observables. In particular, a large set of differential cross section and recoil polarization data is provided by Kelly [159], see Sec. IV A. Figure 4 shows the quality of fit to the Kelly data for a representative kinematical configuration. Except for a few points, the data are described very well. While the Kelly set has a higher χ^2 per datum than displayed in Table I for the full database, i.e., a $\chi^2 \approx 3$ per data point, approximately 1/3 of the χ^2 comes from 1% outliers. Similar characteristics are found in other observables. In cases where the outliers could be attributed to simple errors in digitizing older data sets, they were not included in our fits.

Another large set of data, taken in a dedicated Jefferson Lab experiment [157,180], was analyzed by EBAC (Excited Baryon Analysis Center) in Ref. [121]. The data considered in the EBAC analysis were composed of structure functions $\{\sigma_T + \epsilon\sigma_L, \sigma_{TT}, \sigma_{LT}, \sigma_{LT'}\}$, in contrast with the present fit, where we included the data of Refs. [157,180] via $d\sigma/d\Omega$, $A_{0Y'}$, and $\sigma_{LT'}$ observables. Note also that Ref. [121] covered a smaller kinematical range, but the model used there had fewer free parameters. The present fits versus structure functions are shown in Fig. 5 for $Q^2 = 0.4 \text{ GeV}^2$ and 0.9 GeV^2 . We note that the description of the structure functions does indeed agree with the results of Refs. [157,180] in nearly all cases, apart from a discrepancy in σ_{LT} at $W = 1.14 \text{ GeV}$. This energy

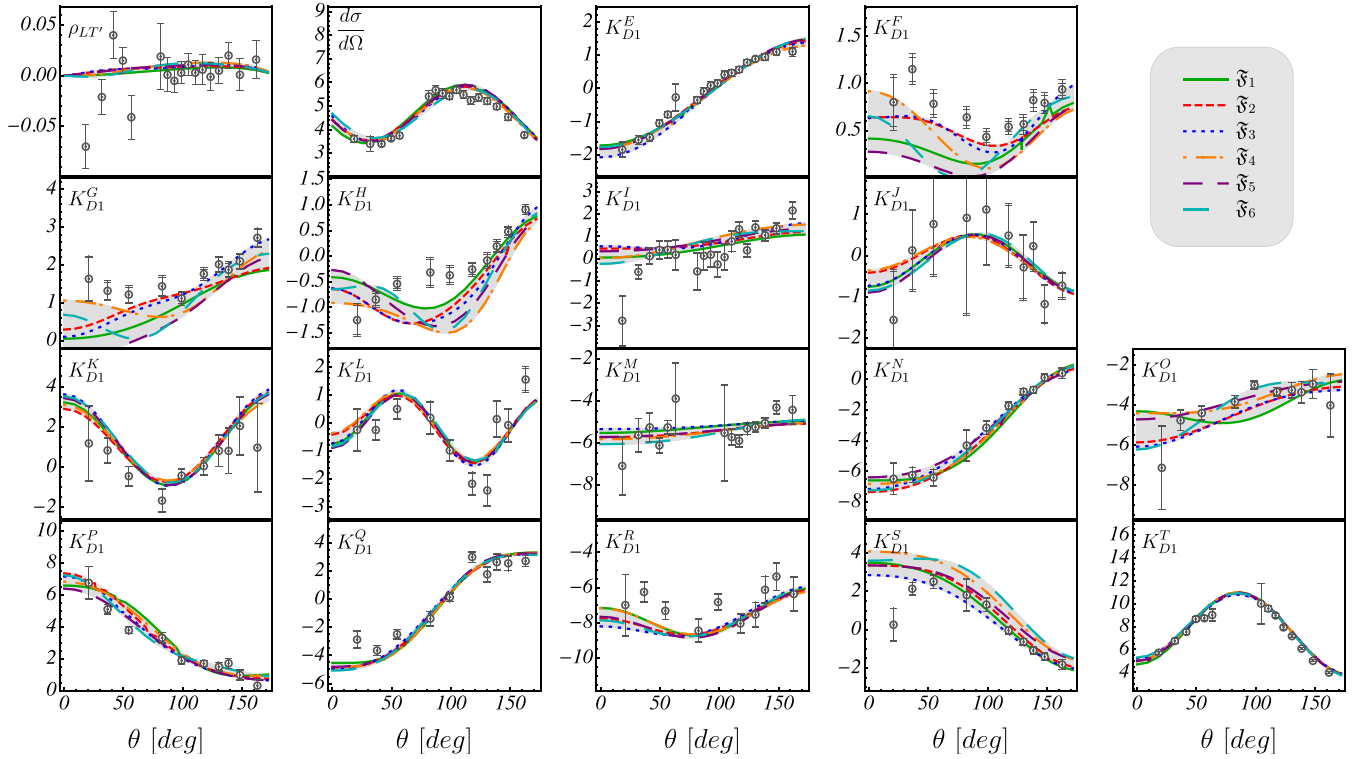


FIG. 4. Comparison of the best fits to the Kelly data [159] (open circles with error bars) at $Q^2 = 1 \text{ GeV}^2$, $W = 1.23 \text{ GeV}$, $\phi = 15^\circ$ in the $\pi^0 p$ channel. Different curves correspond to various fit strategies, representing systematic uncertainty of our approach. Note that $\rho_{LT'}$ is unitless, while others are given in $[\mu\text{b/sr}]$. In this and subsequent figures, the shading between the curves is included to guide the eye.

lies only 10 MeV above the lower limit of data considered in this study and structure functions were not used in the fits directly.

With the sets of model parameters determined, we are now able to plot the E , L , and M multipoles as a function of Q^2 and W . This is demonstrated in a set of plots for some fixed typical values of Q^2 and W . In particular, in Figs. 8, 9, and 10 of Appendix 4 we have evaluated the obtained multipoles at fixed energies corresponding to either the Breit-Wigner mass or real part of the pole position associated with the $\Delta(1232)$, $N(1440)$, and $N(1535)$; $W = 1230$, 1380 , and 1535 MeV , respectively. We observe that, in many cases, the otherwise very weakly constrained longitudinal multipoles are indeed restricted by coupling them to the electric ones at the pseudothreshold point. Then, for two values $Q^2 = 0.2 \text{ GeV}^2$ and 1.0 GeV^2 , we demonstrate the full set of multipoles, up to $J = 2+$, in Figs. 11 and 12, respectively. Results of the MAID2007 analysis [92] are displayed there for comparison (open circles refer to the MAID2007 energy-dependent solution evaluated at a discrete set of W and Q^2). Note that also near the photon point the MAID2007 [92] and JüBo [148], analyses show sizable differences in some photoproduction multipoles.

Examples of multipoles dominated by nucleon resonances are depicted in Fig. 6. The $\Delta(1232)$ and $N(1520)$ display canonical resonance behavior while the η threshold cusp is evident for the $N(1535)$. While the $\Delta(1232)$ shape disappears at higher Q^2 , the $N(1535)$ disappears slower. The “profile” of the enigmatic Roper resonance, $N(1440)$, shows a nontrivial

Q^2 behavior, including zeros for real and imaginary parts. A better quantitative understanding of the Q^2 dependence of resonance will be facilitated by an upcoming analysis at the resonance poles. But it is already reassuring that, e.g., the multipole of the Roper resonance exhibits zeros similar to the pertinent resonance helicity coupling from experiment [2,34,128,219,220] that is predicted by theoretical approaches [31].

An extended plot of the $M_{1+}^{3/2}$ multipole is given in Fig. 7 with a comparison to the MAID values. All fits agree with rather small uncertainties. Note that the first vertical line corresponds to a pseudothreshold point, $Q_{\text{PT}-}$, at which condition (2.18) and Siegert’s condition (2.19) are implemented by construction, whereas the results of MAID2007 [92] are restricted to $Q^2 \geq 0$. While we expect the very large imaginary part of the multipole to be similar in all fits, including MAID, we also observe close agreement for the real part.

A direct comparison with the EBAC fit [121] is not possible but the variations in their single- Q^2 fits emphasize the lack of sufficient data constraints in the low- Q^2 region, and a benefit from including the $Q^2 = 0$ values obtained in photoproduction analyses.

V. CONCLUSIONS AND OUTLOOK

In the current paper, we have introduced a phenomenological parametrization of the meson electroproduction multipoles. This parametrization builds upon the latest Jülich-Bonn solution which includes information from a large database

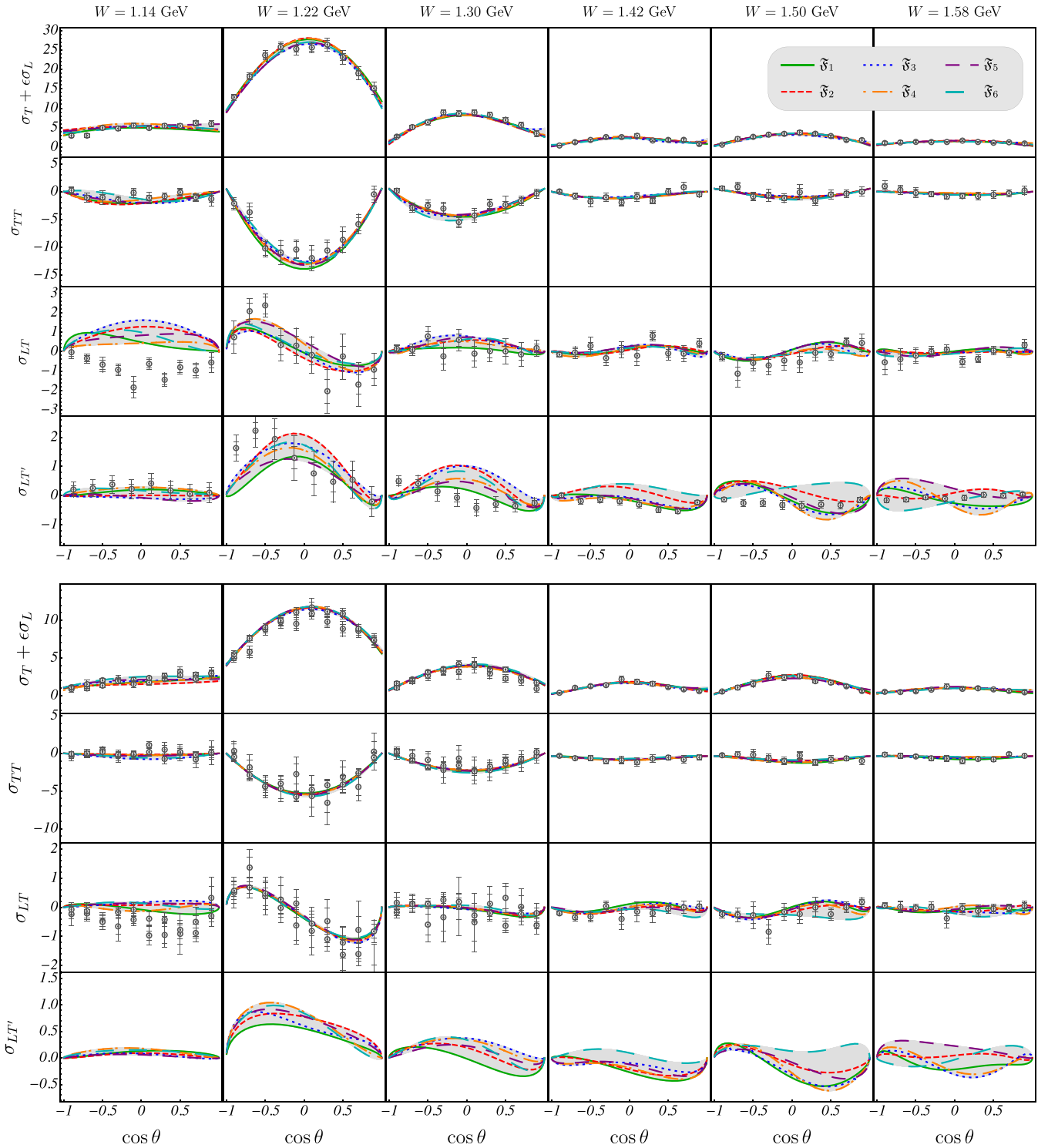


FIG. 5. Best fits to $\pi^0 p$ electroproduction data for $Q^2 = 0.4 \text{ GeV}^2$ (top) and $Q^2 = 0.9 \text{ GeV}^2$ (bottom) compared with the structure functions from Refs. [157,180], shown by open circles with statistical and sum of statistical and systematic error bars, respectively. All units are $[\mu\text{b}/\text{sr}]$. Different curves correspond to various fit strategies, representing systematic uncertainty of our approach.

on meson-baryon scattering and meson photoproduction and takes into account constraints from unitarity, analyticity, and chiral symmetry. This approach is extended to the electroproduction sector by parametrizing the Q^2 dependence with a general analytic form incorporating constraints from Siegert's

theorem and (pseudo)threshold behavior of production amplitudes. Additionally, form factors are included to ensure the falloff of multipoles at large Q^2 .

Overall, multipoles of up to D waves are fit with respect to 209 parameters to reproduce world pion electroproduction

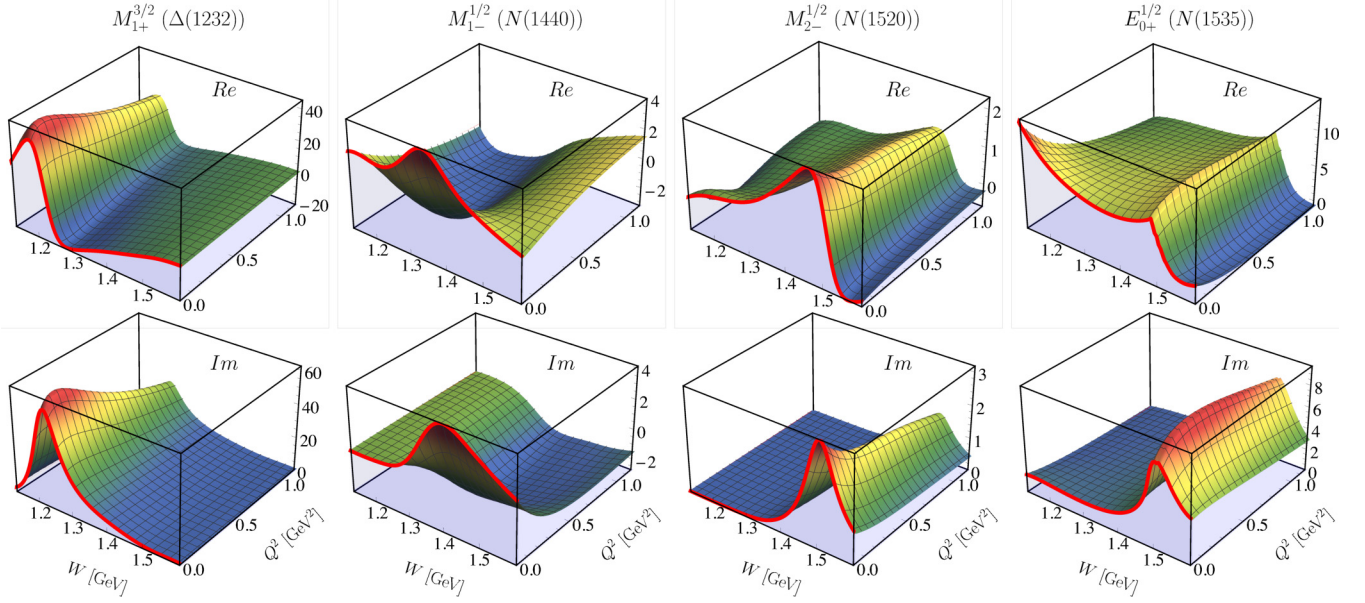


FIG. 6. Selected results (representative fit $\mathfrak{F}_1$) for multipoles associated with $\Delta(1232)$, $N(1440)$, $N(1520)$, and $N(1535)$, respectively. At $Q^2 = 0 \text{ GeV}^2$ the solution is constrained by pion-induced and photoproduction data via the Jülich-Bonn model (red line), while the extension into $Q^2 > 0$ is facilitated by the current parametrization. All in units of mfm.

data over a large range of $1.13 \text{ GeV} < W < 1.6 \text{ GeV}$ and $Q^2 < 6 \text{ GeV}^2$. This range is similar to that of Ref. [122] but we include more polarization observables. We found a good description ($\chi_{\text{dof}}^2 = 1.69\text{--}1.81$) of an extensive database ($\approx 10^5$ data) including observables of polarized and unpolarized types. Additionally, we have provided an uncertainty estimate on obtained multipoles by exploring several fitting scenarios. Taking these uncertainties into account, the predicted multipoles agree qualitatively with those of the previous MAID2007 analysis [92]. This is a nontrivial result keeping in mind the large variety of fit strategies employed in our study, with substantially different initial parameter sets.

In parallel to the present study, a website has been developed [215] utilizing a SQL database, a more modern web framework (Django), and an interactive graphics front end (Plotly), serving as a platform to compare different analyses against the existing data. Presently included models are from the Jülich-Bonn fits (JüBo2017), describing photoproduction

of pions, kaons, and etas, as well as MAID2007 [92], ETA-MAID [93,95], and KAON-MAID [221] for photo- and electroproduction of pions, etas, and kaons. A present version of our electroproduction fits is currently being incorporated.

We plan to extend the formalism to include ηN and $K\Lambda$ final states. While technically straightforward, the further expansion of the parameter space may need to be reassessed using, e.g., model selection techniques as done in Ref. [222]. Furthermore, a combined fit of pion-induced photo- and electroproduction data is planned, which will allow for a more reliable extraction of the helicity couplings of resonances.

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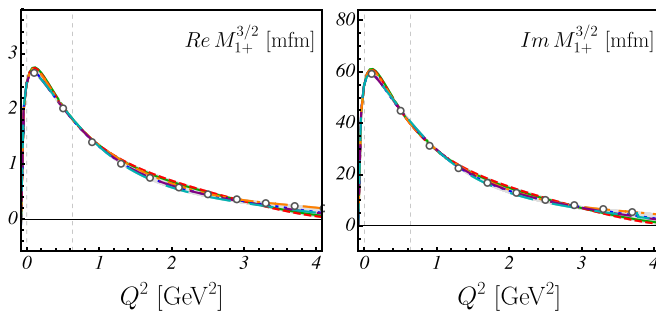


FIG. 7. $M_{1+}^{3/2}$ multipole at $W = 1230 \text{ MeV}$ for six best fits (color coding as in Fig. 4). Results of MAID2007 analysis [92] are given by open circles for comparison only. Dashed vertical lines show pseudothreshold, photon point, and $Q^2(\omega = 0)$, respectively.

TABLE II. Conversion of Kelly observables [159] to the SAID [213] notation. The last column shows the implementation in terms of the helicity amplitudes as employed in this work, while $\xi = \omega/(Q^2)^{1/2}$.

	SAID/current notation	Kelly notation	Expression in helicity amplitudes (H_i)
1	K_{D1}^A	RL	$(H_5^2 + H_6^2)/\xi^2$
2	K_{D1}^B	RL(n)	$\text{Im}(-H_6 H_5^*)2/(\xi^2 \sin \theta)$
3	K_{D1}^C	RT	$(H_1^2 + H_2^2 + H_3^2 + H_4^2)/2$
4	K_{D1}^D	RT(n)	$\text{Im}(+H_3 H_1^* + H_4 H_2^*)/\sin \theta$
5	K_{D1}^E	RLT	$\text{Re}(+H_5 H_1^* - H_5 H_4^* + H_6 H_2^* + H_6 H_3^*)/(\sqrt{2}\xi \sin \theta)$
6	K_{D1}^F	RLT(n)	$\text{Im}(-H_2 H_5^* - H_3 H_5^* + H_1 H_6^* - H_4 H_6^*)/(\sqrt{2}\xi)$
7	K_{D1}^G	RLT(l)	$\text{Im}(+H_1 H_5^* + H_4 H_3^* + H_2 H_6^* - H_3 H_6^*)/(\sqrt{2}\xi \sin \theta)$
8	K_{D1}^H	RLT(t)	$\text{Im}(+H_2 H_5^* - H_3 H_5^* - H_1 H_6^* - H_4 H_6^*)/(\sqrt{2}\xi)$
9	K_{D1}^I	RLT(h)	$\text{Im}(-H_1 H_5^* + H_4 H_3^* - H_2 H_6^* - H_3 H_6^*)/(\sqrt{2}\xi \sin \theta)$
10	K_{D1}^J	RLT(hn)	$\text{Re}(-H_5 H_2^* - H_5 H_3^* + H_6 H_1^* - H_6 H_4^*)/(\sqrt{2}\xi)$
11	K_{D1}^K	RLT(hl)	$\text{Re}(-H_5 H_1^* - H_5 H_4^* - H_6 H_2^* + H_6 H_3^*)/(\sqrt{2}\xi \sin \theta)$
12	K_{D1}^L	RLT(ht)	$\text{Re}(-H_5 H_2^* + H_5 H_3^* + H_6 H_1^* + H_6 H_4^*)/(\sqrt{2}\xi)$
13	K_{D1}^M	RTT	$\text{Re}(-H_1 H_4^* + H_2 H_3^*)/(\sin \theta)^2$
14	K_{D1}^N	RTT(n)	$\text{Im}(-H_2 H_1^* - H_4 H_3^*)/\sin \theta$
15	K_{D1}^O	RTT(l)	$\text{Im}(+H_4 H_1^* - H_3 H_2^*)/(\sin \theta)^2$
16	K_{D1}^P	RTT(t)	$\text{Im}(+H_2 H_1^* - H_4 H_3^*)/\sin \theta$
17	K_{D1}^Q	RTT(hl)	$\text{Re}(-H_1 H_1^* - H_2 H_2^* + H_3 H_3^* + H_4 H_4^*)/2$
18	K_{D1}^R	RTT(ht)	$\text{Re}(+H_1 H_3^* + H_2 H_4^*)/\sin \theta$
19	K_{D1}^S	RL+RT(n)	$(H_5^2 + H_6^2)/\xi^2 - \text{Im}(H_3^* H_1 + H_4^* H_2)/\sin(\theta)$
20	K_{D1}^T	RL+RT	$(H_1^2 + H_2^2 + H_3^2 + H_4^2)/2 + (H_5^2 + H_6^2)/\xi^2$

APPENDIX

1. Blatt-Weisskopf barrier-penetration factors

For $\ell = 0, \dots, 5$, the Blatt-Weisskopf barrier-penetration factors [168,169] are explicitly given by

$$\begin{aligned}
 B_0(r) &= 1, \\
 B_1(r) &= r/\sqrt{1+r^2}, \\
 B_2(r) &= r^2/\sqrt{9+3r^2+r^4}, \\
 B_3(r) &= r^3/\sqrt{225+45r^2+6r^4+r^6}, \\
 B_4(r) &= r^4/\sqrt{11025+1575r^2+135r^4+10r^6+r^8}, \\
 B_5(r) &= r^5/\sqrt{893025+99225r^2+6300r^4+315r^6+15r^8+r^{10}}.
 \end{aligned} \tag{A1}$$

2. Definition of K_{1D} observables

Table II shows the connection of the helicity amplitudes to the Kelly observables.

3. Photoproduction kernel in the JüBo model

In the JüBo approach the photoproduction kernel $V_{\mu\gamma}$ in Eq. (3.4) is parametrized by the quantities $\gamma_{\gamma;i}^c$ and α^{NP} which are constructed with energy-dependent polynomials P^P and P^{NP} [145],

$$\alpha_{\mu\gamma}^{\text{NP}}(p, W) = \frac{\tilde{\gamma}_{\mu}^a(p)}{\sqrt{m}} P_{\mu}^{\text{NP}}(W) \text{ and } \gamma_{\gamma;i}^c(W) = \sqrt{m} P_i^P(W). \tag{A2}$$

The vertex function $\tilde{\gamma}_{\mu}^a$ is equal to $\gamma_{\mu;i}^a$ but independent of the resonance number i . The polynomials P are explicitly given by

$$P_i^P(W) = \sum_{j=1}^{l_i} g_{i,j}^P \left(\frac{W - E_s}{m} \right)^j e^{-\lambda_i^P(W - E_s)}, \tag{A3}$$

$$P_{\mu}^{\text{NP}}(W) = \sum_{j=0}^{l_{\mu}} g_{\mu,j}^{\text{NP}} \left(\frac{W - W_s}{m} \right)^j e^{-\lambda_{\mu}^{\text{NP}}(W - W_s)}. \tag{A4}$$

Here, $g^{\text{P(NP)}}$ and $\lambda^{\text{P(NP)}} > 0$ are multipole-dependent free parameters that are fit to data. The upper limits of the summation l_i and l_{μ} are chosen as demanded by the data. In Ref. [148], which is used as input for the present study, $l_i, l_{\mu} \leq 3$ is

sufficient to achieve a good fit result. The expansion point E_s is chosen as $W_s = 1077$ MeV in order to be close to the πN threshold.

4. Multipoles for fixed W and Q^2

Figures 8 to 10 show the all relevant multipoles for the fixed total energy W , whereas Figs. 11 to 12 show those for the fixed virtuality Q^2 .

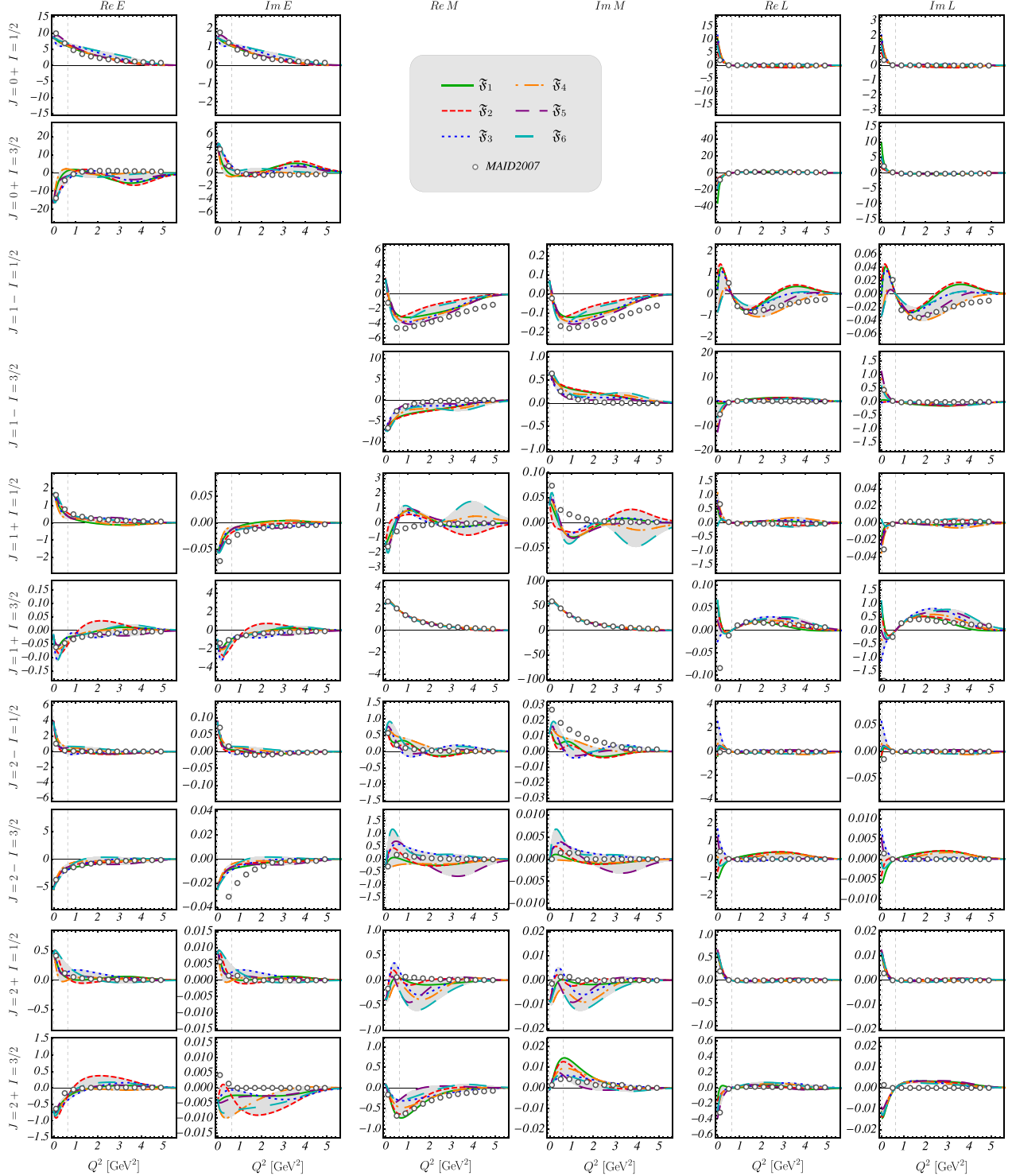


FIG. 8. Fit results for multipoles in [mfm] at $W = 1.230$ GeV as a function of Q^2 . Different curves correspond to various fit strategies, representing systematic uncertainty of our approach—shading between the curves is included to guide the eye. Results of MAID2007 analysis [92] are shown by open circles for comparison. Dashed vertical lines show virtualities corresponding to $q = 0$ and $\omega = 0$, respectively.

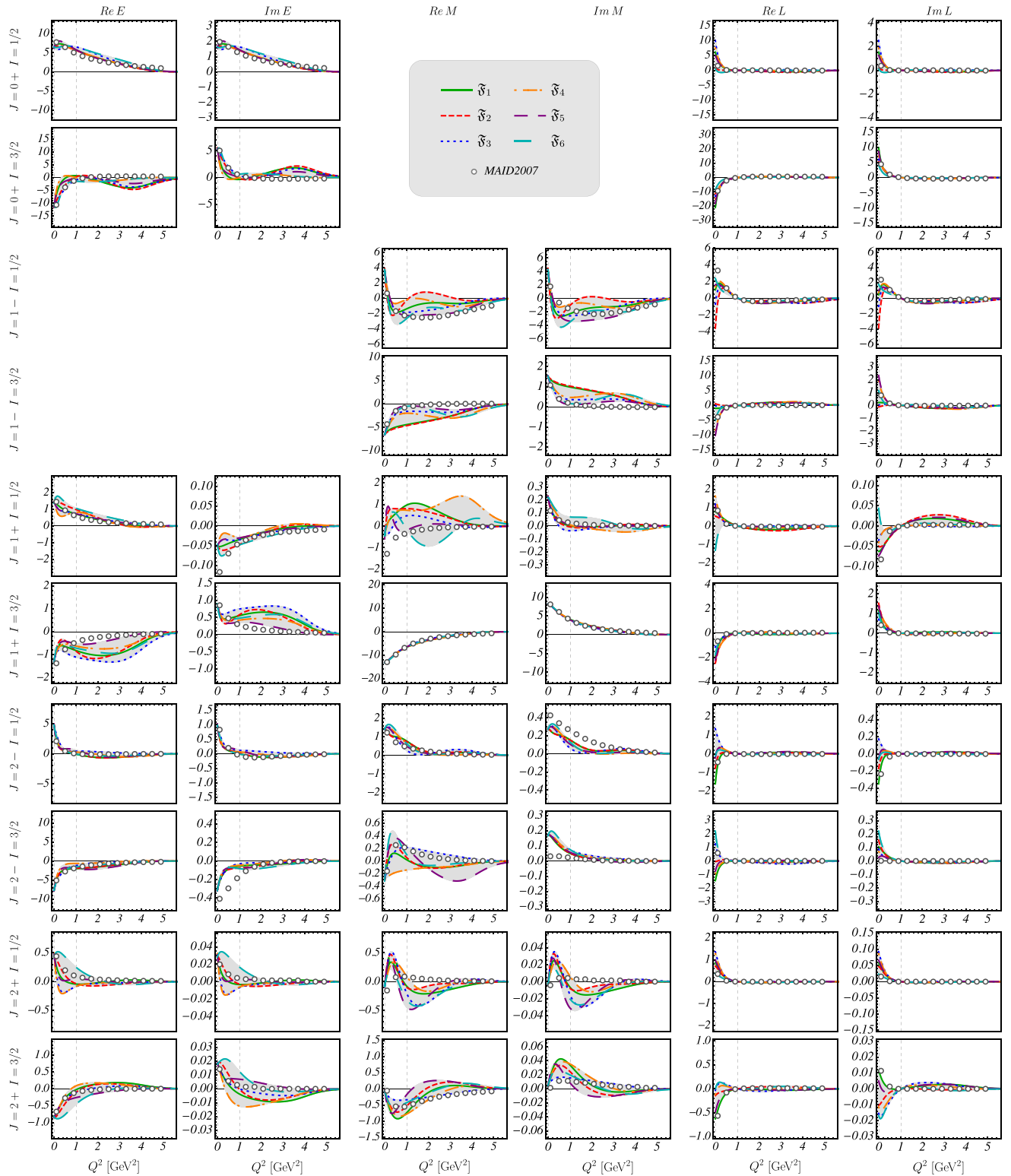


FIG. 9. Fit results for multipoles in [mfm] at $W = 1.380$ GeV as a function of Q^2 . Different curves correspond to various fit strategies, representing systematic uncertainty of our approach—shading between the curves is included to guide the eye. Results of MAID2007 analysis [92] are shown by open circles for comparison. Dashed vertical lines show virtualities corresponding to $q = 0$ and $\omega = 0$, respectively.

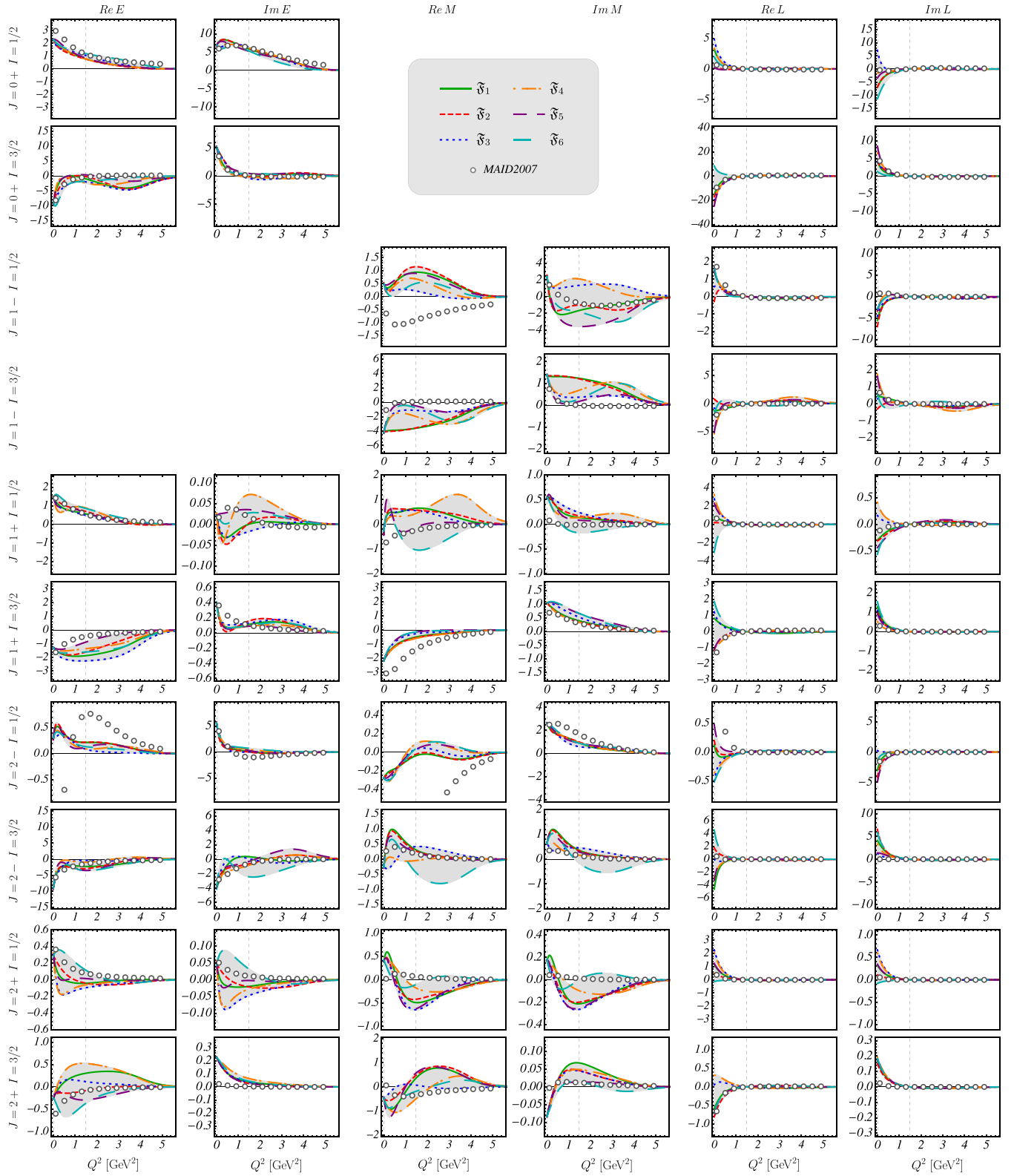


FIG. 10. Fit results for multipoles in [mfm] at $W = 1.535$ GeV as a function of Q^2 . Different curves correspond to various fit strategies, representing systematic uncertainty of our approach—shading between the curves is included to guide the eye. Results of MAID2007 analysis [92] are shown by open circles for comparison. Dashed vertical lines show virtualities corresponding to $q = 0$ and $\omega = 0$, respectively.

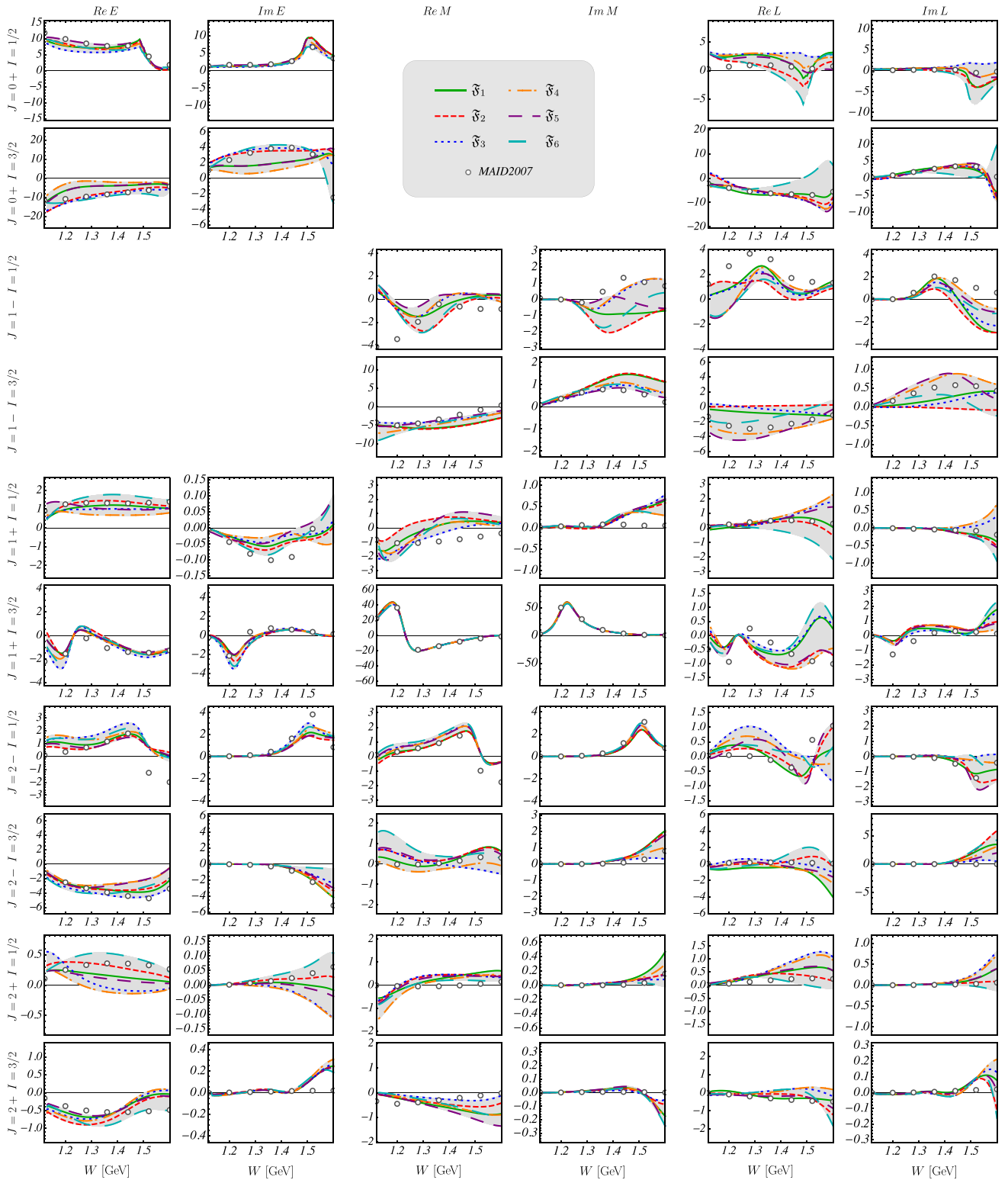


FIG. 11. Fit results for multipoles in [mfm] at $Q^2 = 0.2 \text{ GeV}^2$ as a function of W . Different curves correspond to various fit strategies, representing systematic uncertainty of our approach—shading between the curves is included to guide the eye. Results of MAID2007 analysis [92] are shown by open circles for comparison.

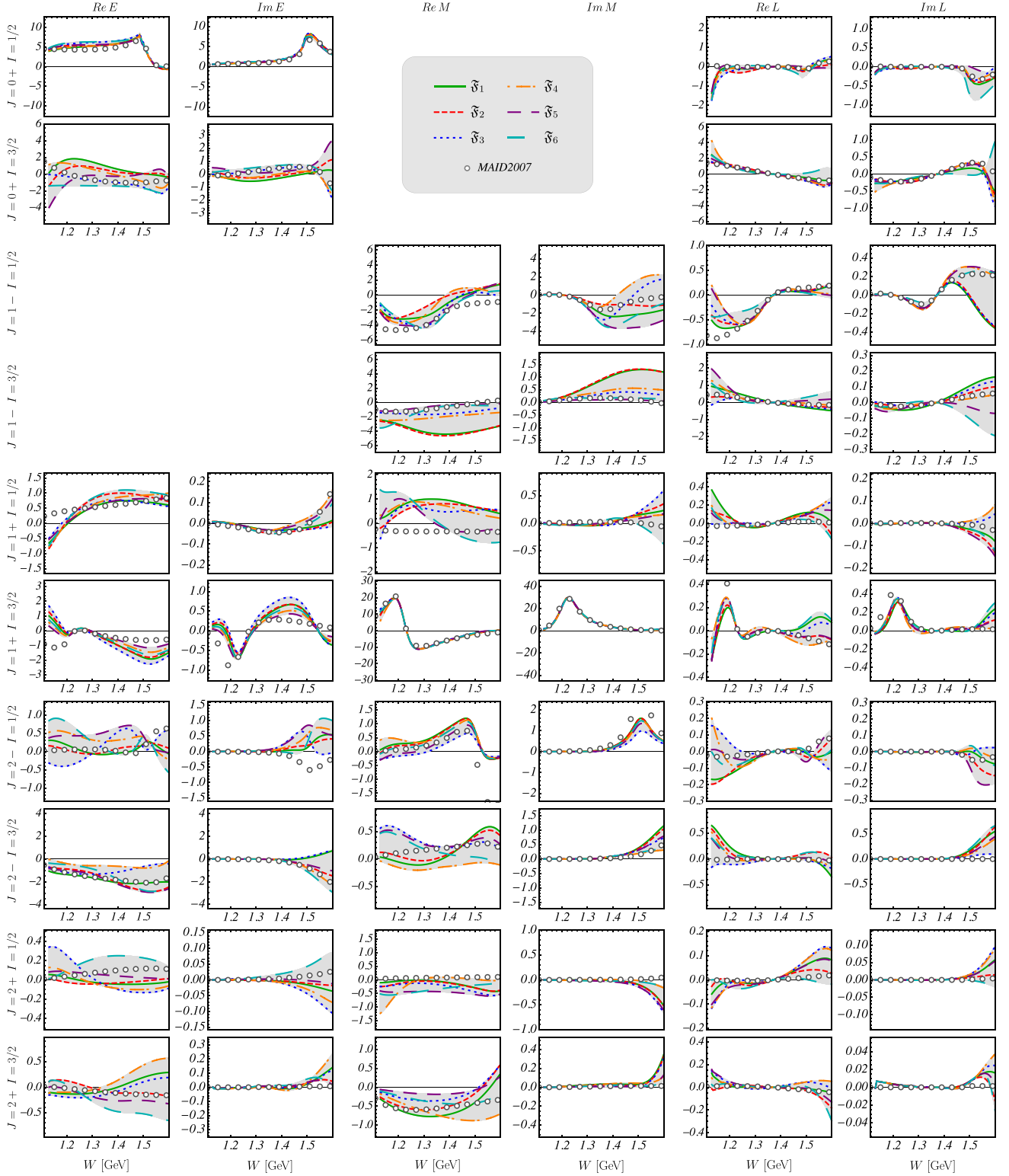


FIG. 12. Fit results for multipoles in [mfm] at $Q^2 = 1.0 \text{ GeV}^2$ as a function of W . Different curves correspond to various fit strategies, representing systematic uncertainty of our approach—shading between the curves is included to guide the eye. Results of MAID2007 analysis [92] are shown by open circles for comparison.

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