Journal of Mathematical Physics

Time evolution of two harmonic oscillators with cross-Kerr interactions

Cite as: J. Math. Phys. **61**, 032102 (2020); https://doi.org/10.1063/1.5121397
Submitted: 24 July 2019 • Accepted: 07 February 2020 • Published Online: 02 March 2020

David Edward Bruschi

View Online Export Citation CrossMark

ARTICLES YOU MAY BE INTERESTED IN

Semiclassical theories as initial value problems
Journal of Mathematical Physics **61**, 032301 (2020); https://doi.org/10.1063/1.5122782

The asymptotic behavior of the time fractional Schrödinger equation on Hilbert space
Journal of Mathematical Physics **61**, 031501 (2020); https://doi.org/10.1063/1.5142272

Mathematics of the classical and the quantum
Journal of Mathematical Physics **61**, 082101 (2020); https://doi.org/10.1063/1.5135015

© 2020 Author(s).

Time evolution of two harmonic oscillators with cross-Kerr interactions

Cite as: J. Math. Phys. 61, 032102 (2020); doi: 10.1063/1.5121397
Submitted: 24 July 2019 • Accepted: 7 February 2020 •
Published Online: 2 March 2020

View Online Export Citation CrossMark

David Edward Bruschi[1,2,a)]

AFFILIATIONS

[1] Central European Institute of Technology (CEITEC), Brno University of Technology, Purkyňova 123, 621 00 Brno, Czech Republic
[2] Institute for Quantum Optics and Quantum Information - IQOQI Vienna, Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria

[a)] Author to whom correspondence should be addressed: david.edward.bruschi@gmail.com

ABSTRACT

We study the time evolution of two coupled quantum harmonic oscillators interacting through nonlinear optomechanical-like Hamiltonians that include cross-Kerr interactions. We employ techniques developed to decouple the time-evolution operator and obtain the analytical solution for the time evolution of the system. We apply these results to obtain explicit expressions of a few quantities of interest. Our results do not require approximations and therefore allow us to study the nature and implications of the full nonlinearity of the system. As a potential application, we show that it is possible to greatly increase the population of phonons using a suitable combination of cubic and cross-Kerr interactions.

Published under license by AIP Publishing. https://doi.org/10.1063/1.5121397

I. INTRODUCTION

The understanding of the dynamics of quantum systems is paramount to deepen our understanding of the laws of Nature. Given an arbitrary Hamiltonian, it is, in general, impossible to obtain a full and analytical expression for the time evolution of the system it describes. Typically, approximations or *ad hoc* methods are required, leaving open the question of the existence of a general framework or set of tools to address the problem systematically.

In recent years, specific techniques have been developed to tackle the problem for Gaussian states of bosonic systems interacting through quadratic Hamiltonians.[1–3] This has led to the development of a mathematical framework aimed at obtaining the full time evolution of the system using the power of the covariance matrix formalism.[4] Interestingly, these techniques do not apply to this class of states and transformations only, but it can be generalized to arbitrary states and arbitrary Hamiltonians. This novel approach has led to very interesting preliminary results, where complete and analytical solutions of the time evolution of the system were found for optomechanical-like Hamiltonians of arbitrary numbers of modes of light interacting with arbitrary numbers of mechanical resonators.[5,6] This has allowed, for example, for a quantitative analysis of the nonlinear character of an optomechanical system.[7]

In this work, we take another step forward and study the time evolution of a system of two bosonic mode interactions via an optomechanical-like Hamiltonian with cross-Kerr terms. Such a Hamiltonian does not model a realistic optomechanical system since it does not take into account external drive, loss or dissipation. However, this Hamiltonian can be used to model ideal quantum systems interacting through nonlinear terms that can potentially be designed in realistic implementations where loss and decoherence can be (greatly) reduced. Such systems might include optomechanical systems with appropriately tuned cross-Kerr terms,[8] pulsed optomechanical systems where the measurement times are comparable or smaller than decoherence times,[9] vibrational modes of atoms trapped in periodic cavity field potentials,[10] as well as superconducting circuits that simulate (quadratic) optomechanical interactions.[11]

We are able to find an analytical solution to the time evolution which we use to compute the expectation value of relevant quantities as a function of time, such as the number of "phonons" in the system, as well as the mixedness induced by the interaction in the state of the resonator's subsystem. We specialize our results to a few scenarios of interest, which model light–matter interaction, that can be exploited for sensing and gravimetry,[12] as well as tests of fundamental physics.[13] In addition, our results are connected to work aimed at finding solutions to the evolution induced by cubic, or higher, order Hamiltonians that are entirely obtained as combinations of polynomial powers of quadrature—or creation and annihilation—operators.[14] Ultimately, our work can aid in deepening our understanding of the mathematical framework of quantum mechanics, one of our two pillars of modern science.

The paper is structured as follows: In Sec. II, we introduce the general framework necessary for our work. In Sec. III, we decouple the time evolution of the system for the general case. In Sec. IV, we specialize to scenarios of more practical interest. Finally, we briefly give concluding remarks in Sec. VI.

II. GENERAL FRAMEWORK

We start by introducing the Hamiltonian and the tools necessary to this work. More details about the computations are left to Appendixes A–D. Extensive introduction to the frameworks mentioned below is left to the literature.

A. Hamiltonian

In this work, we consider the general Hamiltonian $\hat{H} = \hat{H}_0(t) + \hat{H}_{\mathrm{I}}(t)$, where $\hat{H}_0 := \hbar\,\omega_{\mathrm{c}}(t)\hat{a}^\dagger\hat{a} + \hbar\,\omega_{\mathrm{m}}\,\hat{b}^\dagger\hat{b}$ is the *potentially time-dependent* free Hamiltonian and

$$\hat{H}_{\mathrm{I}}(t) = \hbar\,g_1^{(+)}(t)\,\hat{a}^\dagger\hat{a}\,\hat{B}_+ + \hbar\,g_1^{(-)}(t)\,\hat{a}^\dagger\hat{a}\,\hat{B}_- + \hbar\,g_2^{(+)}(t)\,\hat{a}^\dagger\hat{a}\,\hat{B}_+^{(2)} + \hbar\,g_2^{(-)}(t)\,\hat{a}^\dagger\hat{a}\,\hat{B}_-^{(2)} + 2\,\hbar\,g_2'(t)\,\hat{a}^\dagger\hat{a}\,\hat{b}^\dagger\hat{b}, \tag{1}$$

and we have defined $\hat{B}_+ := \hat{b}^\dagger + \hat{b}$, $\hat{B}_- := i(\hat{b}^\dagger - \hat{b})$, $\hat{B}_+^{(2)} := \hat{b}^{\dagger 2} + \hat{b}^2$, and $\hat{B}_-^{(2)} := i(\hat{b}^{\dagger 2} - \hat{b}^2)$ for notational convenience. The operators $\hat{B}_\pm$ correspond to the quadrature operators, while the operators $\hat{B}_\pm^{(2)}$ are directly related to the square of the quadrature operators (modulo an additional term proportional to $\hat{b}^\dagger\,\hat{b}$). We choose to retain our notation because it is more natural to the techniques used in this work. An advantage of expression (1) is that it allows us to consider each cross-Kerr term independently from each other.

The Hamiltonian (1) is a formal extension of well-known Hamiltonians that can model light and matter interaction, e.g., optomechanical Hamiltonians.[15] The main difference is that we consider an "ideal" case here, namely, we do not include an external (laser) drive and we assume that the system is lossless and noiseless. In this sense, the system considered here does not model realistic optomechanical systems. Nevertheless, it provides a platform for studying cross-Kerr interactions, which can be of practical interest with the advance of technological control.

B. Dimensionless dynamics

To understand which are the relevant dimensionless parameters that govern the dynamics of the system, we introduce dimensionless quantities and rescale the Hamiltonian (1). This is achieved rescaling all frequencies and time by the reference frequency ω_{m}. Therefore, the laboratory time t will become $\tau = \omega_{\mathrm{m}} t$, where τ is the new dimensionless time. The couplings in the Hamiltonian are subsequently relabeled as follows:

$$\tilde{g}_1^{(\pm)}(\tau) = g_1^{(\pm)}(\omega_{\mathrm{m}}\,t)/\omega_{\mathrm{m}}, \qquad \tilde{g}_2^{(\pm)}(\tau) = g_2^{(\pm)}(t\omega_{\mathrm{m}})/\omega_{\mathrm{m}}, \qquad \tilde{g}_2'(\tau) = g_2'(t\omega_{\mathrm{m}})/\omega_{\mathrm{m}}. \tag{2}$$

We also rescale the optical frequency to $\tilde{\omega}_{\mathrm{c}}(\tau) := \omega_{\mathrm{c}}(\omega_{\mathrm{m}}\,t)/\omega_{\mathrm{m}}$ and the Hamiltonian by $\hbar$. This implies that we use the rescaled Hamiltonian $\hat{H}(\tau) = \hat{H}_0(\tau) + \hat{H}_{\mathrm{I}}(\tau)$, where we have $\hat{H}_0(\tau) := \tilde{\omega}_{\mathrm{c}}(\tau)\,\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ and

$$\hat{H}_{\mathrm{I}}(\tau) = \tilde{g}_1^{(+)}(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{B}_+ + \tilde{g}_1^{(-)}(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{B}_- + 2\,\tilde{g}_2'(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{b}^\dagger\hat{b} + \tilde{g}_2^{(+)}(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{B}_+^{(2)} + \tilde{g}_2^{(-)}(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{B}_-^{(2)}. \tag{3}$$

With this dimensionless Hamiltonian, it will be easier to understand the interplay between the different scales of the problem and to perform approximations.

C. Solving the dynamics

Here, we outline the tools needed to solve the dynamics generated by (3). We refer the reader to Appendixes A and B for detailed calculations.

The time-evolution operator of a quantum system with time dependent Hamiltonian $\hat{H}(t)$ reads

$$\hat{U}(t) = \overleftarrow{\mathcal{T}}\,\exp\left[-\frac{i}{\hbar}\int_0^t dt'\,\hat{H}(t')\right], \tag{4}$$

where $\overleftarrow{\mathcal{T}}$ is the time ordering operator.[3] This expression simplifies to $U(t) = \exp[-\frac{i}{\hbar}\,\hat{H}\,t]$ when the Hamiltonian $\hat{H}$ is time independent. However, we are interested in Hamiltonians with time dependent parameters.

Published under license by AIP Publishing

Any Hamiltonian can be cast in the form $\hat{H}(t) = \sum_n \hbar g_n(t)\hat{G}_n$, where the $\hat{G}_n$ are time-independent Hermitian operators, the $g_n(t)$ are time-dependent real functions, and the choice of $\hat{G}_n$ is not unique.

In general, one aims to recast (4) in the form

$$\hat{U}(t) = \prod_n \hat{U}_n(t), \tag{5}$$

where we have defined $\hat{U}_n := \exp[-iF_n(t)\hat{G}_n]$ and the real functions $F_n(t)$ are, in general, time dependent. If such an expression exists, we say that the time-evolution operator has been *decoupled*. The functions $F_n(t)$ can be found using the techniques developed in the literature[3] and are determined solely by the parameters of the Hamiltonian (An alternative approach was also attempted in a related approach.[16]). The order of the operators in (5) is not unique. A different order will change the form of the functions $F_n(t)$ but not the expectation value of measurable quantities. A more detailed outline of these techniques can be found in Appendix A.

D. Initial state

The last ingredient in this work is the choice of the initial state of the system. We assume that the optical mode is initially in a coherent state $|\mu_c\rangle$, while the mechanical is in a thermal state $\hat{\rho}_m(T)$. These states are defined by $\hat{a}|\mu_c\rangle = \mu_c|\mu_c\rangle$ and $\hat{\rho}_m(T) = \sum_n \frac{\tanh^{2n} r}{\cosh^2 r}|n\rangle\langle n|$, where the squeezing parameter r is related to the temperature T by the expression $\tanh r = \exp[-\frac{\hbar\,\omega_m}{2\,k_B\,T}]$.

Therefore, the initial state $\hat{\rho}(0)$ reads

$$\hat{\rho}(0) = |\mu_c\rangle\langle\mu_c| \otimes \hat{\rho}_m(T). \tag{6}$$

We choose to introduce $N_a(0) = |\mu|^2$ and $N_b(0) = \sinh^2 r$ as the initial number of excitation of modes $\hat{a}$ and $\hat{b}$, respectively. This state is a good approximation for the initial state of realistic systems, i.e., of optomechanical systems.[15]

III. DECOUPLING OF A NONLINEAR TIME-DEPENDENT OPTOMECHANICAL CROSS-KERR HAMILTONIAN

The first aim of this work is to obtain an analytical solution to the decoupled time-evolution operator (5), given our Hamiltonian (3). Techniques exist already that can be used to tackle the decoupling of time evolution operators in the way that is suitable for us.[3] These are the techniques we will employ here.

A. Decoupling algebra of the nonlinear Hamiltonian

Decoupling of the Hamiltonian (3) can be done using a choice of the Hermitian operators $\hat{G}_n$ (see Appendix A).

The first step is to write

$$\hat{U}(\tau) = e^{-i\int_0^\tau d\tau' \tilde{\omega}_c(\tau')\hat{a}^\dagger\,\hat{a}}\, e^{-i\,\hat{\theta}_2\,\hat{b}^\dagger\,\hat{b}}\,\hat{U}_I(\tau), \tag{7}$$

where we have introduced $\hat{\theta}_2 = \hat{\theta}_2(\hat{a}^\dagger\,\hat{a}) := \tau + 2\int_0^\tau d\tau'\,\tilde{g}_2'(\tau')\,\hat{a}^\dagger\hat{a}$ for notational convenience. Note that $\hat{\theta}_2$ is a time-dependent operator-function of the Hermitian operator $\hat{a}^\dagger\hat{a}$. We can think of $\hat{U}_I(\tau)$ as the time evolution operator of the interaction picture.

We now need to consider the operator $\hat{U}_I(\tau) := \overleftarrow{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_0^\tau \hat{H}_I(\tau)\right]$ induced by the Hamiltonian $\hat{H}_I(\tau) = \hat{H}_{sq}(\tau) + \hat{H}_{OM}(\tau)$, where

$$\begin{aligned}\hat{H}_{sq}(\tau) &:= \hat{a}^\dagger\hat{a}\left[\tilde{g}_2^{(+)}(\tau)\,\cos(2\,\hat{\theta}_2) - \tilde{g}_2^{(-)}(\tau)\,\sin(2\,\hat{\theta}_2)\right]\hat{B}_+^{(2)} + \hat{a}^\dagger\hat{a}\left[\tilde{g}_2^{(+)}(\tau)\,\sin(2\,\hat{\theta}_2) + \tilde{g}_2^{(-)}(\tau)\,\cos(2\,\hat{\theta}_2)\right]\hat{B}_-^{(2)},\\ \hat{H}_{OM}(\tau) &:= \hat{a}^\dagger\hat{a}\left[\tilde{g}_1^{(+)}(\tau)\,\cos(\hat{\theta}_2) - \tilde{g}_1^{(-)}(\tau)\,\sin(\hat{\theta}_2)\right]\hat{B}_+ + \hat{a}^\dagger\hat{a}\left[\tilde{g}_1^{(+)}(\tau)\,\sin(\hat{\theta}_2) + \tilde{g}_1^{(-)}(\tau)\,\cos(\hat{\theta}_2)\right]\hat{B}_-.\end{aligned} \tag{8}$$

We then split the evolution as $\hat{U}_I(\tau) = \hat{U}_{sq}(\tau)\hat{U}_{OM}(\tau)$, where we have defined

$$\begin{aligned}\hat{U}_{sq} &:= \overleftarrow{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_0^\tau d\tau'\,\hat{H}_{sq}(\tau')\right],\\ \hat{U}_{OM} &:= \overleftarrow{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_0^\tau d\tau'\,\hat{U}_{sq}^\dagger\,\hat{H}_{OM}(\tau')\,\hat{U}_{sq}\right].\end{aligned} \tag{9}$$

We need to supplement this ansatz with a second one, namely, with the action of the two-mode squeezing-like operator $\hat{U}_{sq}$ on the operators $\hat{b}$ and $\hat{b}^\dagger$. We have

$$\hat{U}_{sq}^\dagger\,\hat{b}\,\hat{U}_{sq} = \hat{\alpha}\,\hat{b} + \hat{\beta}\,\hat{b}^\dagger, \tag{10}$$

Published under license by AIP Publishing

where $\hat{\alpha} = \hat{\alpha}(\hat{a}^\dagger\hat{a})$ and $\hat{\beta} = \hat{\beta}(\hat{a}^\dagger\hat{a})$ are the Bogoliubov coefficients. These coefficients are *time-dependent* functions of the operator $\hat{a}^\dagger\hat{a}$. They satisfy the Bogoliubov identities $\hat{\alpha}\,\hat{\alpha}^\dagger - \hat{\beta}\,\hat{\beta}^\dagger = \mathbb{1}$ and $\hat{\alpha}\,\hat{\beta}^\dagger - \hat{\beta}\,\hat{\alpha}^\dagger = 0$.

This allows us to obtain

$$\hat{U}_{\text{sq}}^\dagger\,\hat{H}_{\text{OM}}(\tau)\,\hat{U}_{\text{sq}} = \hat{a}^\dagger\hat{a}\,\Re\Big[\tilde{g}_1(\tau)\,\hat{E}_2\,(\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\Big]\,\hat{B}_+ - i\,\hat{a}^\dagger\hat{a}\,\Im\Big[\tilde{g}_1(\tau)\,\hat{E}_2\,(\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\Big]\,\hat{B}_-, \tag{11}$$

where we have defined $\hat{E}_2 = \hat{E}_2(\hat{a}^\dagger\hat{a}) := \cos(\hat{\theta}_2) + i\,\sin(\hat{\theta}_2)$, the coupling $\tilde{g}_1(\tau) := \tilde{g}_1^{(+)}(\tau) + i\,\tilde{g}_1^{(-)}(\tau)$ and $\Re(\hat{O}) := \frac{1}{2}\big[\hat{O} + \hat{O}^\dagger\big]$ for convenience of notation.

Given all of the above, we can finally obtain

$$\hat{U}(\tau) = e^{-i\left[\int_0^\tau d\tau'\,\tilde{\omega}_c(\tau')\hat{a}^\dagger\hat{a} - \hat{F}^{(2)}(\hat{a}^\dagger\hat{a})^2 + \hat{\theta}_2\hat{b}^\dagger\hat{b}\right]}\,\hat{U}_{\text{sq}}\,e^{-i\int_0^\tau d\tau'\,\hat{a}^\dagger\hat{a}\Re\left[\tilde{g}_1(\tau')\hat{E}_2(\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\right]\hat{B}_+}\,e^{i\int_0^\tau d\tau'\,\hat{a}^\dagger\hat{a}\Im\left[\tilde{g}_1(\tau')\hat{E}_2(\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\right]\hat{B}_-}, \tag{12}$$

which is the main expression for the decoupled time-evolution operator in this work.

To complete our main result (12), we require the expression of $\hat{F}^{(2)}$ and of the Bogoliubov coefficients $\hat{\alpha}$ and $\hat{\beta}$. The expression for $\hat{F}^{(2)}$ is readily found as

$$\hat{F}^{(2)} = 2\int_0^\tau d\tau'\,\Im\Big[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\Big]\int_0^{\tau'} d\tau''\,\Re\Big[\tilde{g}_1(\tau'')\,\hat{E}_2\,(\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\Big]. \tag{13}$$

This leaves us with the task of computing the Bogoliubov coefficients $\hat{\alpha}$ and $\hat{\beta}$ in order to obtain a fully analytical understanding of our system.

B. The action of the single-mode squeezing operator

We have noted that, in general, the action of $\hat{U}_{\text{sq}}$ on the operators $\hat{b}$ has the form (10). Ideally, we would like to have an analytical expression for the functional form of $\hat{\alpha}$ and $\hat{\beta}$ in terms of $\hat{a}^\dagger\hat{a}$, and of the couplings of the system. Although this is not possible in general,[17] we proceed to construct two uncoupled differential equations that relate the derivatives of $\hat{\alpha}$ and $\hat{\beta}$ to the couplings $\tilde{g}_2^{\pm}(\tau)$ and $\tilde{g}_2'(\tau)$.

The expression for the single mode squeezing is given in (10). In Appendix B, we show that

$$\begin{aligned}\hat{\alpha} &= \hat{p}_{11},\\ \hat{\beta} &= -i\,\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\,\tilde{\chi}_2(\tau')\,e^{2\,i\,(\phi_2+\hat{\theta}_2)}\,\hat{p}_{11}^\dagger,\end{aligned} \tag{14}$$

where we have introduced the modulus $\tilde{\chi}_2(\tau) := \sqrt{\tilde{g}_2^{(+)2}(\tau) + \tilde{g}_2^{(-)2}(\tau)}$, the angle ϕ_2 through $\tan(2\,\phi_2) = \tilde{g}_2^{(-)}/\tilde{g}_2^{(+)}$, and the operator-function $\hat{p}_{11} = \hat{p}_{11}\big(\int_0^\tau d\tau'\,\tilde{\chi}_2(\tau')\big)$, which satisfies the second-order differential equation

$$\tilde{\chi}_2\,\ddot{\hat{p}}_{11} + i\,\frac{d}{d\tau}(\phi_2 + \hat{\theta}_2)\,\dot{\hat{p}}_{11} - \tilde{\chi}_2\,(a^\dagger\hat{a})^2\,\hat{p}_{11} = 0. \tag{15}$$

The derivative here is with respect to $y(\tau) := \int_0^\tau d\tau'\tilde{\chi}_2(\tau')$. These differential equations have to be supplemented by the initial conditions $\hat{p}_{11}(0) = \mathbb{1}$ and $\dot{\hat{p}}_{11}(0) = 0$.

The evolution of the initial state $\hat{\rho}(0)$ is obtained by the usual Heisenberg equation $\hat{\rho}(\tau) = \hat{U}(\tau)\,\hat{\rho}(0)\,\hat{U}^\dagger(\tau)$. The full expression is not illuminating and we do not print it here. However, we can ask what is the expression for the time evolution of the mode operators $\hat{a}$ and $\hat{b}$. These expressions allow us, in principle, to compute the expectation value of most quantities of interest.

We define $\hat{a}(\tau) := \hat{U}^\dagger(\tau)\hat{a}\hat{U}(\tau)$ and $\hat{b}(\tau) := \hat{U}^\dagger(\tau)\,\hat{b}\,\hat{U}(\tau)$, and, using expression (12), we find

$$\hat{b}(\tau) = \hat{E}_2^\dagger\Big\{\hat{\alpha}\,\hat{b} + \hat{\beta}\,\hat{b}^\dagger - i\,\hat{a}^\dagger\hat{a}\,\hat{\alpha}\,\hat{I} + i\,\hat{a}^\dagger\hat{a}\,\hat{\beta}\,\hat{I}^\dagger\Big\}. \tag{16}$$

Here, $\hat{I} = \hat{I}(\hat{a}^\dagger\,\hat{a}) := \int_0^\tau d\tau'\big[\tilde{g}_1(\tau')\,\hat{E}_2\,\hat{\alpha}^\dagger + \tilde{g}_1^*(\tau')\,\hat{E}_2^\dagger\,\hat{\beta}\big]$ for notational convenience. The expression for $\hat{a}(\tau)$ depends on the explicit functional form of $\hat{F}^{(2)}$, $\Im\big[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\big]$, and $\Re\big[\tilde{g}_1(\tau'')\,\hat{E}_2\,(\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\big]$ in terms of $\hat{a}^\dagger\hat{a}$. This can be computed once the scenario of interest has been determined.

The number $N_{\text{b}}(\tau) := \langle\hat{U}^\dagger(\tau)\,\hat{b}^\dagger\,\hat{b}\,\hat{U}(\tau)\rangle$ of "phonons" at any time τ can be computed using (16), and it reads

$$N_{\text{b}}(\tau) = N_{\text{b}}(0) + (1 + 2N_{\text{b}}(0))e^{-|\mu|^2}\sum_n \frac{|\mu|^{2n}}{n!}|\beta_n|^2 + e^{-|\mu|^2}\sum_{n=1}\frac{n|\mu|^{2n}}{(n-1)!}\Big[|I_n|^2 + 2|\beta_n|^2|I_n|^2 - 2\Re\big(\alpha_n\beta_n^* I_n^2\big)\Big]. \tag{17}$$

In the expression of (17), the subscript n means within each operator we need to replace $n \to \hat{a}^\dagger\hat{a}$.

Published under license by AIP Publishing

IV. APPLICATIONS: TIME EVOLUTION WITH CROSS-KERR TERMS

We are now in the position to study the time evolution induced by the Hamiltonian (12) within some specific cross-Kerr scenarios. We will be able to obtain some analytical expressions for meaningful quantities that encode the full nonlinear character of the system.

A. Cross-Kerr without squeezing

Here, we start with the scenario where $\tilde{g}_2^{(\pm)}(\tau) = \tilde{\chi}_2(\tau) = 0$. This implies that the only cross-Kerr term that does *not* vanish is the term $\tilde{g}_2'(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{b}^\dagger\hat{b}$ in (3). Since $\tilde{\chi}_2(\tau) = 0$, we can immediately see that $\hat{p}_{11} = \mathbb{1}$ and therefore $\hat{\alpha} = \mathbb{1}$ and $\hat{\beta} = 0$. Some algebra allows us to find the expression for (12) in this case, which reads

$$\hat{U}(t) = e^{-i\left[\int_0^\tau d\tau'\tilde{\omega}_{\mathrm{c}}(\tau')\hat{a}^\dagger\,\hat{a} - \hat{F}^{(2)}\,(\hat{a}^\dagger\hat{a})^2 + \hat{\theta}_2\,\hat{b}^\dagger\,\hat{b}\right]}\, e^{-i\int_0^\tau d\tau'\,\hat{a}^\dagger\hat{a}\left[\tilde{g}_1^{(+)}(\tau')\,\cos(\hat{\theta}_2) - \tilde{g}_1^{(-)}(\tau')\,\sin(\hat{\theta}_2)\right]\hat{B}_+}\, e^{i\int_0^\tau d\tau'\,\hat{a}^\dagger\hat{a}\left[\tilde{g}_1^{(+)}(\tau')\,\sin(\hat{\theta}_2) + \tilde{g}_1^{(-)}(\tau')\,\cos(\hat{\theta}_2)\right]\hat{B}_-}, \tag{18}$$

where $\hat{F}^{(2)}$ reads

$$\hat{F}^{(2)} = 2\int_0^\tau d\tau'\left[\tilde{g}_1^{(+)}(\tau')\,\sin(\hat{\theta}_2) + \tilde{g}_1^{(-)}(\tau')\,\cos(\hat{\theta}_2)\right]\int_0^{\tau'} d\tau''\left[\tilde{g}_1^{(+)}(\tau'')\,\cos(\hat{\theta}_2) - \tilde{g}_1^{(-)}(\tau'')\,\sin(\hat{\theta}_2)\right]. \tag{19}$$

Note that, for $\tilde{g}_2'(\tau) = \tilde{g}_1^{(-)} = 0$, we recover the results found in the literature, as expected.[5]

Expression (18) allows us to find

$$\hat{b}(\tau) = e^{-i\,\hat{\theta}_2}\left[\hat{b} - i\,\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\,\tilde{g}_1(\tau')\,e^{i\,\hat{\theta}_2}\right], \tag{20}$$

where we have omitted the expression for $\hat{a}(\tau)$, which *can* be computed but is not illuminating.

The change $\Delta N_{\mathrm{b}}(\tau) := N_{\mathrm{b}}(\tau) - N_{\mathrm{b}}(0)$ in the number of phonons, given our initial state (6), reads

$$\Delta N_{\mathrm{b}}(\tau) = |\mu|^2\,e^{-|\mu|^2}\sum_n \frac{(n+1)\,|\mu|^{2\,n}}{n!}\,|I_{n+1}|^2, \tag{21}$$

where $I_n = \int_0^\tau d\tau'\tilde{g}_1(\tau')\,\exp\left[i\left(\tau + 2\,n\int_0^\tau d\tau'\,\tilde{g}_2'(\tau')\right)\right]$ for this case.

Finally, we can also compute the mixedness of the reduced state. The calculations can be found in Appendix D. Lengthy algebra, and the use of a similar approach developed in the literature,[6] allows us to find

$$S_N(\hat{\rho}_{\mathrm{m}}(\tau)) = 1 - e^{-2\,|\mu|^2}\sum_{n,n'}\frac{|\mu|^{2\,n+2\,n'}}{n!\,n'!}\,\frac{\exp\left[-\frac{|\Delta_{nn'}|^2}{\cosh r_T}\right]}{\cosh r_T}, \tag{22}$$

where we have defined $\Delta_{nn'} := \int_0^\tau d\tau'\,\tilde{g}_1(\tau')\left(n\,e^{i\,\theta_{2,n}(\tau')} - n'\,e^{\theta_{2,n'}(\tau')}\right)$ and $\theta_{2,n}(\tau) := \tau + 2\,i\,n\int_0^\tau d\tau''\,\tilde{g}_2'(\tau'')$ for convenience of presentation. It is easy to find the result for zero temperature: it is sufficient to set $r_T = 0$ in (22). Note that when $\Delta_{nn'} = 0$, i.e., $\tilde{g}_1(\tau) = 0$, we obtain $S_N(\hat{\rho}_{\mathrm{m}}(\tau)) = S_N(\hat{\rho}_{\mathrm{m}}(0))$ as expected.

B. Cross-Kerr with squeezing and without diagonal term

Here, we consider the simpler scenario when $\tilde{g}_2'(\tau) = 0$. This implies that the cross-Kerr term that *vanishes* is $\tilde{g}_2'(\tau)\,\hat{a}^\dagger\hat{a}\,\hat{b}^\dagger\hat{b}$ in (12). Since $\tilde{g}_2'(\tau) = 0$, the main differential equation (15) reads

$$\tilde{\chi}_2(\tau)\,\ddot{\hat{p}}_{11} - i\left(1 + \frac{d}{d\tau}\phi_2\right)\dot{\hat{p}}_{11} - \tilde{\chi}_2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\hat{p}_{11} = 0. \tag{23}$$

This case requires numerical integration of the differential equation above, since there is no analytical solution for a general form of the coupling $\tilde{\chi}_2(\tau)$. A numerical approach can then enable the use of (12).

C. Cross-Kerr with constant squeezing and constant diagonal term

In this final case, we consider $\tilde{g}_2'(\tau) = \tilde{g}_2'$ and $\tilde{g}_2(\tau) := \tilde{g}_2^{(+)}(\tau) + i\,\tilde{g}_2^{(-)}(\tau) = \tilde{g}_2$ constant. This implies that also $\tilde{\chi}_2$ and ϕ_2 are constant. In turn, this means that (15) reads

$$\tilde{\chi}_2\,\ddot{\hat{p}}_{11} - i\,(1 + 2\,\tilde{g}_2'\,\hat{a}^\dagger\hat{a})\,\dot{\hat{p}}_{11} - \tilde{\chi}_2\,(\hat{a}^\dagger\hat{a})^2\,\hat{p}_{11} = 0, \tag{24}$$

Published under license by AIP Publishing

given that $\hat{\theta}_2 = (1 + 2\,\tilde{g}_2'\,\hat{a}^\dagger\hat{a})\,\tau$ in this scenario. This allows us to find the solution

$$\hat{p}_{11}(\tau) = e^{i\,\hat{K}\,\tau}\left[\cos\big(\hat{\Lambda}\,\tau\big) - i\,\frac{\hat{K}}{\hat{\Lambda}}\,\sin\big(\hat{\Lambda}\,\tau\big)\right], \tag{25}$$

where we have introduced $\hat{K} := 1 + 2\,\tilde{g}_2'\,\hat{a}^\dagger\hat{a}$ and $\hat{\Lambda} := \sqrt{\hat{K}^2 - 4\,\tilde{\chi}_2^2\,(\hat{a}^\dagger\hat{a})^2}$ for convenience of presentation. Note that $\hat{\Lambda}^{-1}\,\sin(\hat{\Lambda}\,\tau)$ is a well-defined operator, since it is the short hand notation for the expression $\hat{\Lambda}^{-1}\,\sin\big(\hat{\Lambda}\,\tau\big) = \big[1 - (\hat{\Lambda}\,\tau)^2/3! + (\hat{\Lambda}\,\tau)^4/5! - \ldots\big]\,\tau$. Also note that $\hat{\theta}_2 = \hat{K}\,\tau$ in our present case. Finally, note that if we were to set $\tilde{\chi}_2 = 0$, we would have $\hat{\Lambda} = \hat{K}$ and we can immediately see that $\hat{p}_{11}(\tau) = \mathbb{1}$, as expected from our results above.

Recalling that $\hat{p}_{11}(\tau)$ in (25) gives us one of the Bogoliubov coefficients directly, i.e., $\hat{\alpha} = \hat{p}_{11}(\tau)$, we can easily find the other coefficient by employing (14), which gives

$$\begin{aligned}\hat{\alpha} &= e^{i\,\hat{K}\,\tau}\left[\cos\big(\hat{\Lambda}\,\tau\big) - i\,\frac{\hat{K}}{\hat{\Lambda}}\,\sin\big(\hat{\Lambda}\,\tau\big)\right],\\ \hat{\beta} &= -2\,i\,\tilde{\chi}_2\,\frac{\hat{a}^\dagger\hat{a}}{\hat{\Lambda}}\,e^{2\,i\,\phi_2}\,e^{i\,\hat{K}\,\tau}\,\sin\big(\hat{\Lambda}\,\tau\big).\end{aligned} \tag{26}$$

It is immediate to check that $\hat{\alpha}\,\hat{\alpha}^\dagger - \hat{\beta}\,\hat{\beta}^\dagger = \mathbb{1}$.

This allows us to obtain an analytical expression for the time evolution operator (12). We first obtain $\hat{F}^{(2)}$, which reads (13), together with an expression for $\Re\big[\tilde{g}_1(\tau)\,\hat{E}_2\,(\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\big]$ and $\Im\big[\tilde{g}_1(\tau)\,\hat{E}_2\,(\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\big]$, whose full expressions can be found in (B16).

We do not need an analytical expression for the decoupled form of the operator $\hat{U}_{\text{sq}}$ for this case, since we have found the Bogoliubov coefficients (26).

This means that our time evolution operator (12) has the explicit expression (B17), which can be specialized to any desired functional expression of the $\tilde{g}_1$ drive.

D. Resonant selective squeezing

Here, we apply our results to a specific case of potential interest for future implementations of interactions of light and matter. We assume that $\tilde{g}_2'(\tau) = \tilde{g}_2'$ is constant and $\tilde{g}_2^{(-)}(\tau) = 0$ and that $\tilde{g}_2^{(+)}(\tau) = \tilde{g}_2 = \tilde{\chi}_2$ is constant. This implies that $\phi_2 = 0$. We also assume that $\tilde{g}_1(\tau) = \tilde{\chi}_1$ constant.

The first observation to make is that the squeezing operation induced by $\hat{H}_{\text{sq}} = \tilde{\chi}_1\,\hat{a}^\dagger\hat{a}\,\hat{B}_+ + (1 + 2\,\tilde{g}_2'\,\hat{a}^\dagger\hat{a})\,\hat{b}^\dagger\hat{b} + \tilde{\chi}_2\,\hat{a}^\dagger\hat{a}\,\hat{B}_+^{(2)}$ can cast in matrix form. This can be done after tracing over the photonic degrees of freedom, which gives us a collection of operators induced by $\hat{H}_{\text{sq},n} = n\,\tilde{\chi}_1\,\hat{B}_+ + (1 + 2\,n\,\tilde{g}_2')\,\hat{b}^\dagger\hat{b} + n\,\tilde{\chi}_2\,\hat{B}_+^{(2)}$. This allows us to study the eigenvalues of the Hamiltonian matrix $\boldsymbol{H}_{\text{sq},n}$ that represents $\hat{H}_{\text{sq},n}$ and is defined by $\hat{H}_{\text{sq},n} = \frac{1}{2}\,\mathbb{X}^\dagger\,\boldsymbol{H}_{\text{sq},n}\,\mathbb{X}$, where $\mathbb{X} = (\hat{b}, \hat{b}^\dagger)^{\text{Tp}}$. A simple calculation tells us that the eigenvalues $\lambda_{n,\pm}$ of the Hamiltonian matrix $\boldsymbol{H}_{\text{sq},n}$ read

$$\lambda_{n,\pm} = 1 + 2\,n\,\tilde{g}_2' \pm 2\,n\,\tilde{\chi}_2. \tag{27}$$

We now note that $\Lambda_n = \sqrt{K_n^2 - 4\,\tilde{\chi}_2^2\,n^2} = \sqrt{\lambda_{n,+}\,\lambda_{n,-}}$. Therefore, for those values of n for whom $\lambda_{n,\pm} < 0$, we have that $\Lambda_n \to i\,\Lambda_n = i\,\sqrt{4\,\tilde{\chi}_2^2\,n^2 - K_n^2}$ and the Bogoliubov coefficients (26), for such n, read

$$\begin{aligned}\alpha_n &= e^{i\,K_n\,\tau}\left[\cosh(\Lambda_n\,\tau) - i\,\frac{K_n}{\Lambda_n}\,\sinh(\Lambda_n\,\tau)\right],\\ \beta_n &= -2\,i\,\tilde{\chi}_2\,\frac{n}{\Lambda_n}\,e^{i\,K_n\,\tau}\,\sinh(\Lambda_n\,\tau).\end{aligned} \tag{28}$$

Note that the condition $\lambda_{n,\pm} < 0$, therefore, depends on the relative magnitude of the couplings with respect to n, and crucially on their sign.

If there exists a particular $n = n_* \in \mathbb{N}$ such that $\Lambda_{n_*} = \sqrt{K_{n_*}^2 - 4\,\tilde{\chi}_2^2\,n_*^2} = 0$, (28) immediately gives us

$$\begin{aligned}\alpha_{n_*} &= e^{i\,\hat{K}\,\tau}\,[1 - 2\,i\,\tilde{\chi}_2\,n_*\,\tau],\\ \beta_{n_*} &= -2\,i\,\tilde{\chi}_2\,n_*\,e^{i\,\hat{K}\,\tau}\,\tau.\end{aligned} \tag{29}$$

We now note the following: when $\tilde{\chi}_2 > \tilde{g}_2'$ the Bogoliubov coefficients (28) apply for each $n > 1/(2\,(\tilde{\chi}_2 - \tilde{g}_2'))$. In case $\tilde{\chi}_2 < \tilde{g}_2'$, there does not exist any $n \in \mathbb{N}$ such that the Bogoliubov coefficients have the behavior (28).

Published under license by AIP Publishing

If it is possible to have $\tilde{\chi}_2 > \tilde{g}'_2$, this allows us immediately to compute I_n, and therefore, the change in phononic population $\Delta N_{\mathrm{b}}(\tau)$ through the general expression (17) reads

$$\Delta N_{\mathrm{b}}(\tau) = 4\,\tilde{\chi}_2^2\,(1+2\,N_{\mathrm{b}}(0))\,e^{-|\mu|^2}\sum_{n=1}^{+\infty}\frac{n\,|\mu|^{2\,n}}{(n-1)!}\,\frac{\mathrm{sh}_n^2}{\Lambda_n^2} + \tilde{\chi}_1^2\,e^{-|\mu|^2}\sum_{n=1}\frac{n\,|\mu|^{2\,n}}{(n-1)!}\,\frac{\mathrm{sh}_n^2 + (K_n \mp 2\,\tilde{\chi}_2\,n)^2\,(\mathrm{ch}_n - 1)^2}{\Lambda_n^2}, \tag{30}$$

where $\mathrm{ch}_n := \cosh(\Lambda_n\,\tau)$ and $\mathrm{sh}_n := \sinh(\Lambda_n\,\tau)$ for simplicity of presentation. Here, the $\mp$ sign corresponds to our choice $\tilde{g}_1 = e^{\frac{i}{4}\,(\pi\mp\pi)}\,\tilde{\chi}_1$, where $\tilde{\chi}_1 \geq 0$.

It is clear from (30) that, when $\tau \gg 1$, the contributions in the sums that for which n give one $\lambda_{n,\pm} < 0$ grow exponentially with time. Each contribution grows as $\sim \exp[\Lambda_n \tau]$, and it is dampened by the $|\mu|^n/(n-1)!$ in front.

V. CONSIDERATIONS

Our results can be applied to any ideal optomechanical system with arbitrary cross-Kerr interaction, which includes as a special case a system of two-harmonic oscillators with only cross-Kerr interactions and no cubic term [that is when $\tilde{g}_1^{(\pm)}(\tau) = 0$].

We would like to make here a few considerations on our results in order to give a context of their applicability and overview of their scope.

(i) First, our results are "general," in the sense that they apply regardless of the specific implementation to be used or the physical and experimental limits on the parameter regimes. In this sense, our work can motivate pushing the limits of studies of existing systems and expanding beyond traditional boundaries on their parameter regimes, since we have provided analytical tools to investigate new areas in the parameter space.

(ii) Our results are "fully nonlinear," which means that the full nonlinearity of the system is taken into account and is responsible for the final results. Linear responses are easy to find and can be obtained by simply setting $|\tilde{g}_2| \ll 1$ and $\tilde{g}'_2 \ll 1$ and retaining terms to first order in these parameters. This approach would be equivalent to employing perturbation theory, and the results would provide nothing new with respect of what has been already studied in the literature in such weak cross-Kerr regimes.

(iii) It might not be possible, at least at the moment, to engineer Hamiltonians of the general form (3). However, our Hamiltonian covers all possible combinations of quartic cross-Kerr terms, which also provides further insight into the algebraic structure of linear operators that act on the Hilbert space.

(iv) Our initial Hamiltonian (3) models an *ideal* system, namely, a system that has no losses and no decoherence. To include such phenomena, one would need to modify the Heisenberg equation $\frac{d}{dt}\hat{\rho} = \frac{i}{\hbar}[\hat{H},\hat{\rho}]$ to the *master equation* $\frac{d}{dt}\hat{\rho} = \frac{i}{\hbar}[\hat{H},\hat{\rho}] + \hat{\mathcal{L}}_{\mathrm{D}}\hat{\rho}$, where $\hat{\mathcal{L}}_{\mathrm{D}}$ is an operator that implements these important processes. That decoherence is important in most systems, such as optomechanical ones,[15], and there are standard techniques to find and work with master equations such as the one just mentioned.[15] In the current study, our goal here was to give the first steps into understanding how the full nonlinear aspects of the ideal cross-Kerr scenario affect important quantities, such as the number of phonons or the mixedness of the subsystems. Potential applications of our results can be to studies that show that cross-Kerr terms can be used to greatly reduce decoherence itself,[8] as well as studies that show that simulations of (quadratic) optomechanical systems in superconducting circuits can be performed in low-decoherence regimes.[11] We leave it to further work to include decoherence and dissipation to the approach presented here.

VI. CONCLUSIONS

We have studied the time evolution induced by a class of Hamiltonians of two interacting quantum harmonic oscillators that include nonlinear optomechanical-like and cross-Kerr interactions, without losses and without decoherence. We employed tools developed to obtain an analytical expression for the time-evolution operator, which, in turn, allowed us to explicitly compute quantities of interest. These include the number expectation value of the mechanical resonator—that is, the number of phonons in the system—and the mixedness of the state of the resonator. Our results are free from approximations and therefore encode the full nonlinear character of the interaction.

These techniques, and the control gained by employing them, can have many applications. For example, similar decoupling techniques have been recently applied to the analysis of correlations within the tripartite coupled bosonic system interacting with quadratic Hamiltonians,[18] which prompted and facilitated a successful experiment investigating the resulting entanglement between the bosonic modes.[19] Among other possible applications of our tools, there is quantum control,[20] understanding of the interplay between the linear and nonlinear characters of quantum mechanical systems[7] and extending current studies of hidden quantum correlations in existing electromechanical measurements.[21]

Finally, we have also identified parameter regimes that allow for a potentially exponential increase in the number of phonons and therefore in the mixedness induced between the two subsystems. This feature can have important applications in the quest of demonstrating quantum aspects of macroscopic systems.[15,22]

Published under license by AIP Publishing

ACKNOWLEDGMENTS

We would like to thank Anja Metelmann, Sorin Paraoanu, Alessio Serafini, and Myung-Joong Hwang for useful comments. D.E.B. acknowledges the University of Vienna and the Erwin Schrödinger Institute (ESI) for hospitality. D.E.B. also acknowledges support from the CEITEC Nano RI.

APPENDIX A: DECOUPLING OF TECHNIQUES FOR TIME EVOLUTION

In this appendix, we outline the general decoupling techniques that we shall be using throughout this work to find a decoupled time-evolution operator generated by the Hamiltonian in (3). These techniques have been developed in previous work.[1,3]

1. Decoupling for arbitrary Hamiltonians

The time evolution operator $\hat{U}(t)$ induced by a Hamiltonian $\hat{H}(t)$ reads

$$\hat{U}(t) = \overleftarrow{\mathcal{T}} \exp\left[-\frac{i}{\hbar}\int_0^t dt'\,\mathcal{H}(t')\right], \tag{A1}$$

where $\overleftarrow{\mathcal{T}}$ is the time-ordering operator.

Any Hamiltonian $\hat{H}(t)$ can be cast in the form $\hat{H}(t) = \sum_n \hbar g_n(t)\hat{G}_n$, where the $\hat{G}_n$ are time independent, Hermitian operators and the $g_n(t)$ are time dependent functions. The choice of $\hat{G}_n$ is not unique.

We say that the time evolution operator (A1) has been *decoupled* if it is possible to obtain the expression

$$\hat{U}(t) = \prod_n \hat{U}_n(t), \tag{A2}$$

where we have defined $\hat{U}_n := \exp[-iF_n(t)\hat{G}_n]$ and the real, time-dependent functions $F_n(t)$ can be obtained as explained below. It has been shown that it is always possible to obtain the decoupling in terms of symplectic matrices when the Hamiltonian is quadratic in the bosonic operators,[3] which can be of great help for practical computations.

The functions $F_n(t)$ are uniquely determined by the coupled, nonlinear, first order differential equations

$$\frac{1}{\hbar}\hat{H} = \dot{F}_1\,\hat{G}_1 + \dot{F}_2\,\hat{U}_1\,\hat{G}_2\,\hat{U}_1^\dagger + \dot{F}_3\,\hat{U}_1\,\hat{U}_2\,\hat{G}_3\,\hat{U}_2^\dagger\,\hat{U}_1^\dagger + \dot{F}_4\,\hat{U}_1\,\hat{U}_2\,\hat{U}_3\,\hat{G}_4\,\hat{U}_3^\dagger\,\hat{U}_2^\dagger\,\hat{U}_1^\dagger + \dots. \tag{A3}$$

This is the general method we have been employed in this work and in previous related one.

Here, we find it convenient to consider the *formally infinite* Lie algebra generated by the following set of Hermitian basis operators,

$$\begin{aligned} &(\hat{a}^\dagger\hat{a})^N, \quad (\hat{a}^\dagger\hat{a})^N\hat{b}^\dagger\hat{b},\\ &(\hat{a}^\dagger\hat{a})^N\,\hat{B}_+, \quad (\hat{a}^\dagger\hat{a})^N\,\hat{B}_-,\\ &(\hat{a}^\dagger\hat{a})^N\hat{B}_+^{(2)}, \quad (\hat{a}^\dagger\hat{a})^N\hat{B}_-^{(2)}, \end{aligned} \tag{A4}$$

which are the maximal extension of those that generate the Hamiltonian (3). Note that there are an infinite of these operators, i.e., there are operators for all $N \in \mathbb{N}$.

2. Decoupling for quadratic Hamiltonians: Continuous variables and covariance matrix formalism

If the Hamiltonian is quadratic in the mode operators, the techniques described in Appendix A 1 have a more powerful representation. Here, we proceed to describe these techniques.

In quantum mechanics, the initial state $\hat{\rho}_\mathrm{I}$ of a system of N bosonic modes with operators $\{\hat{a}_n, \hat{a}_n^\dagger\}$ evolves to a final state $\hat{\rho}_f$ through the standard von Neumann equation $\hat{\rho}_f = \hat{U}\,\hat{\rho}_\mathrm{I}\,\hat{U}^\dagger$, where U implements the transformation of interest, such as time evolution. If the state $\hat{\rho}$ is Gaussian and the Hamiltonian H is quadratic in the operators, it is convenient to introduce the vector $\hat{\mathbb{X}} = (\hat{a}_1, \dots, \hat{a}_N, \hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger)^{\mathrm{Tp}}$, the vector of first moments $d := \langle\hat{\mathbb{X}}\rangle$ and the covariance matrix $\boldsymbol{\sigma}$ defined by $\sigma_{nm} := \langle\{\hat{X}_n, \hat{X}_m^\dagger\}\rangle - 2\langle\hat{X}_n\rangle\langle\hat{X}_m^\dagger\rangle$, where $\{\cdot,\cdot\}$ stands for anticommutator and all expectation values of an operator $\hat{\mathcal{A}}$ are defined by $\langle\hat{\mathcal{A}}\rangle := \mathrm{Tr}(\hat{\mathcal{A}}\,\hat{\rho})$. In this language, the canonical commutation relations read $[\hat{X}_n, \hat{X}_m^\dagger] = i\,\Omega_{nm}$, where the $2N \times 2N$ matrix $\boldsymbol{\Omega}$ is known as the symplectic form.[4] We then note that, while arbitrary states of bosonic modes are, in general, characterized by an infinite amount of degrees of freedom, a Gaussian state is uniquely determined by its first and second moments, d_n and σ_{nm}, respectively.[4] Furthermore, quadratic, i.e., linear, unitary transformations, such as Bogoliubov transformations, preserve the Gaussian character of the Gaussian state and can always be represented by a $2N \times 2N$ symplectic matrix $\boldsymbol{S}$ that preserves the symplectic form, i.e., $\boldsymbol{S}^\dagger\boldsymbol{\Omega}\,\boldsymbol{S} = \boldsymbol{\Omega}$.

Published under license by AIP Publishing

The von Neumann equation can be translated in this language to the simple equation $\boldsymbol{\sigma}_f = \boldsymbol{S}\,\boldsymbol{\sigma}_{\rm I}\boldsymbol{S}^\dagger$ for the second moments and $d_f = \boldsymbol{S} d_{\rm i}$ for the first moments, which shifts the problem of usually untreatable operator algebra to simple $2N \times 2N$ matrix multiplication. Finally, Williamson's theorem guarantees that any $2N \times 2N$ Hermitian matrix, such as the covariance matrix $\boldsymbol{\sigma}$, can be decomposed as $\boldsymbol{\sigma} = \boldsymbol{S}^\dagger \boldsymbol{\nu}_\oplus \boldsymbol{S}$, where $\boldsymbol{S}$ is an appropriate symplectic matrix, the diagonal matrix $\boldsymbol{\nu}_\oplus = \mathrm{diag}(\nu_1, \ldots, \nu_N, \nu_1, \ldots, \nu_N)$ is known as the Williamson form of the state and $\nu_n := \coth(\frac{2\,\hbar\,\omega_n}{k_B\,T}) \geq 1$ are the symplectic eigenvalues of the state.[23]

Williamson's form $\boldsymbol{\nu}_\oplus$ contains information about the local and global mixedness of the state of the system.[4] The state is pure when $\nu_n = 1$ for all n and is mixed otherwise. As an example, the thermal state $\boldsymbol{\sigma}_{th}$ of a N-mode bosonic system is simply given by its Williamson form, i.e., $\boldsymbol{\sigma}_{th} = \boldsymbol{\nu}_\oplus$.

The time evolution operator has a symplectic representation $\boldsymbol{S}(t) = \overleftarrow{\mathcal{T}} \exp[\boldsymbol{\Omega} \int_0^t dt' \boldsymbol{H}(t')]$, where the matrix $\boldsymbol{H}$ is defined through $\hat{H} = \frac{\hbar}{2}\, \mathbb{X}^\dagger\, \boldsymbol{H}\, \mathbb{X}$, and the decoupled ansatz (A2) has the form

$$\boldsymbol{S} = \prod_{n=1}^{N\,(2N+1)} \boldsymbol{S}_n, \tag{A5}$$

where we have introduced the matrices $\boldsymbol{S}_n. := \exp[F_n(t) \boldsymbol{\Omega}\, \boldsymbol{G}_n]$ defined $\boldsymbol{G}_n$, which are obtained from $\hat{G}_n = \frac{1}{2}\, \mathbb{X}^\dagger\, \boldsymbol{G}_n\, \mathbb{X}$, and the real time-dependent functions $F_n(t)$ that are the *same* as they would be obtained with the technique above.

The real, time dependent functions $F_n(\tau)$ can be obtained by solving the following system of coupled nonlinear first order differential equations:

$$\boldsymbol{H} = \dot{F}_1\, \boldsymbol{G}_1 + \dot{F}_2\, \boldsymbol{S}_1^\dagger\, \boldsymbol{G}_2\, \boldsymbol{S}_1 + \dot{F}_3\, \boldsymbol{S}_1^\dagger\, \boldsymbol{S}_2^\dagger\, \boldsymbol{G}_3\, \boldsymbol{S}_2\, \boldsymbol{S}_1 + \dot{F}_4\, \boldsymbol{S}_1^\dagger\, \boldsymbol{S}_2^\dagger\, \boldsymbol{S}_3^\dagger\, \boldsymbol{G}_4\, \boldsymbol{S}_3\, \boldsymbol{S}_2\, \boldsymbol{S}_1 + \ldots. \tag{A6}$$

This is the matrix version of the operator differential equations (A3) for quadratic Hamiltonians, which reduces the problem of operator algebra to matrix multiplication.

APPENDIX B: DECOUPLING OF THE SQUEEZING HAMILTONIAN

Here, we employ the techniques of Appendix A to decouple the operator $\hat{U}_{\rm sq} = \overleftarrow{\mathcal{T}} \exp\left[-\frac{i}{\hbar}\int_0^\tau d\tau'\, \hat{H}_{\rm sq}(\tau')\right]$ induced by the squeezing Hamiltonian $\hat{H}_{\rm sq}(\tau)$, which we reprint here

$$\hat{H}_{\rm sq}(\tau) := \hat{a}^\dagger \hat{a}\left[\tilde{g}_2^{(+)}(\tau)\, \cos(2\,\hat{\theta}_2) - \tilde{g}_2^{(-)}(\tau)\, \sin(2\,\hat{\theta}_2)\right] \hat{B}_+^{(2)} + \hat{a}^\dagger \hat{a}\left[\tilde{g}_2^{(+)}(\tau)\, \sin(2\,\hat{\theta}_2) + \tilde{g}_2^{(-)}(\tau)\, \cos(2\,\hat{\theta}_2)\right] \hat{B}_-^{(2)}. \tag{B1}$$

1. Preliminaries

We start by defining $\mathbb{X} := (\hat{b}, \hat{b}^\dagger)^{\rm Tp}$ an noting that we can write

$$\mathbb{X}' = \hat{U}_{\rm sq}^\dagger\, \mathbb{X}\, \hat{U}_{\rm sq} = \begin{pmatrix} \hat{U}_{\rm sq}^\dagger\, \hat{b}\, \hat{U}_{\rm sq} \\ \hat{U}_{\rm sq}^\dagger\, \hat{b}^\dagger\, \hat{U}_{\rm sq} \end{pmatrix} = \boldsymbol{S}_{\rm sq}(\tau)\, \mathbb{X}, \tag{B2}$$

where the 2×2 symplectic matrix $\boldsymbol{S}_{\rm sq}(\tau)$ is the symplectic representation of $\hat{U}_{\rm sq}$ and satisfies $\boldsymbol{S}_{\rm sq}^\dagger(\tau)\, \boldsymbol{\Omega}\, \boldsymbol{S}_{\rm sq}(\tau) = \boldsymbol{\Omega}$. Here, $\boldsymbol{\Omega} = \mathrm{diag}(-i, i)$ is the symplectic form for one mode. The matrix $\boldsymbol{S}_{\rm sq}(\tau)$, therefore, has the expression $\boldsymbol{S}_{\rm sq}(\tau) = \overleftarrow{\mathcal{T}}\, \exp[\boldsymbol{\Omega} \int_0^\tau d\tau'\, \boldsymbol{H}_{\rm sq}(\tau')]$, where $\boldsymbol{H}_{\rm sq}(\tau)$ is obtained from $\hat{H}_{\rm sq} = \frac{1}{2}\, \mathbb{X}^\dagger\, \tilde{\boldsymbol{H}}_{\rm sq}\, \mathbb{X}$, and it has the explicit form

$$\tilde{\boldsymbol{H}}_{\rm sq} = 2\, \hat{a}^\dagger \hat{a} \begin{pmatrix} 0 & \tilde{g}_2(\tau)\, e^{2\,i\,\hat{\theta}_2} \\ \tilde{g}_2^*(\tau)\, e^{-2\,i\,\hat{\theta}_2} & 0 \end{pmatrix}. \tag{B3}$$

Here, we have defined $\tilde{g}_2(\tau) = \tilde{\chi}_2(\tau)\, \exp[2\,i\,\phi_2] := \tilde{g}_2^{(+)}(\tau) + i\,\tilde{g}_2^{(-)}(\tau)$, and therefore, $\tilde{\chi}_2(\tau) = \sqrt{\tilde{g}_2^{(+)2}(\tau) + \tilde{g}_2^{(-)2}(\tau)} \geq 0$ and also $\tan(2\,\phi_2) = \tilde{g}_2^{(-)}/\tilde{g}_2^{(+)}$.

Given the above, it follows that through simple algebra, we can obtain

$$\boldsymbol{S}_{\rm sq}(\tau) = \overleftarrow{\mathcal{T}}\, \exp\left[-2\,i\,\hat{a}^\dagger \hat{a} \int_0^\tau d\tau'\, \tilde{\chi}_2(\tau') \begin{pmatrix} 0 & e^{2\,i\,(\phi_2+\hat{\theta}_2)} \\ -e^{-2\,i\,(\phi_2+\hat{\theta}_2)} & 0 \end{pmatrix}\right]. \tag{B4}$$

Published under license by AIP Publishing

2. Solving the matrix time-ordered exponential

Here, we wish to find a formal expression for (B4). We start by noticing that, if we wrote down the time ordered exponential $\boldsymbol{S}_{\text{sq}}(\tau) := \overleftarrow{\mathcal{T}}\exp\big[-i\,\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\boldsymbol{K}(\tau')\big]$ in terms of its definition, we would obtain

$$\boldsymbol{S}_{\text{sq}}(\tau) = \overleftarrow{\mathcal{T}}\exp\left[-2\,i\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\tilde{\chi}_2(\tau')\begin{pmatrix}0 & e^{2\,i(\phi_2+\hat{\theta}_2)}\\ -e^{-2\,i(\phi_2+\hat{\theta}_2)} & 0\end{pmatrix}\right] = \boldsymbol{P} - i\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\boldsymbol{K}(\tau')\boldsymbol{P}(\tau'), \tag{B5}$$

where the matrix $\boldsymbol{K}(\tau)$ is defined as

$$\boldsymbol{K}(\tau) := 2\,\tilde{\chi}_2(\tau)\begin{pmatrix}0 & e^{2\,i\,(\phi_2+\hat{\theta}_2)}\\ -e^{-2\,i\,(\phi_2+\hat{\theta}_2)} & 0\end{pmatrix}, \tag{B6}$$

and the *diagonal* matrix $\boldsymbol{P}(\tau)$ is the object that we need to compute now. Note that it is straightforward to check that $\boldsymbol{P}(\tau)$ is diagonal.

We use the fact that

$$\frac{d}{d\tau}\overleftarrow{\mathcal{T}}\,\exp\left[-i\,\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\,\boldsymbol{K}(\tau')\right] = -i\,\hat{a}^\dagger\hat{a}\,\boldsymbol{K}(\tau)\,\overleftarrow{\mathcal{T}}\,\exp\left[-i\,\hat{a}^\dagger\hat{a}\int_0^\tau d\tau'\,\boldsymbol{K}(\tau')\right] \tag{B7}$$

to find the equation

$$-(\hat{a}^\dagger\hat{a})^2\,\boldsymbol{K}\int_0^\tau d\tau'\,\boldsymbol{K}\,\boldsymbol{P} = \dot{\boldsymbol{P}}, \tag{B8}$$

where now the explicit dependence on τ is dropped for convenience of presentation.

Since $\boldsymbol{K}$ is invertible [for all cases *except* when $\tilde{\chi}_2(\tau) = 0$, which implies $\boldsymbol{P} = \mathbb{1}$], we can manipulate this equation and obtain, after some algebra,

$$\ddot{\boldsymbol{P}} - \begin{pmatrix}\dot{\tilde{\chi}}_2(\tau)/\tilde{\chi}_2(\tau) + 2\,i\,\dfrac{d}{d\tau}(\phi_2+\hat{\theta}_2) & 0\\ 0 & \dot{\tilde{\chi}}_2(\tau)/\tilde{\chi}_2(\tau) - 2\,i\,\dfrac{d}{d\tau}(\phi_2+\hat{\theta}_2)\end{pmatrix}\dot{\boldsymbol{P}} - 4\,\tilde{\chi}_2^2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\boldsymbol{P} = 0. \tag{B9}$$

We can now solve the four differential equations contained in (B9), two of which are trivial and read $\hat{P}_{12} = \hat{P}_{21} = 0$, which read

$$\begin{aligned}\ddot{\hat{P}}_{11} - \left(\dot{\hat{p}}_2(\tau)/\tilde{\chi}_2(\tau) + 2\,i\,\frac{d}{d\tau}(\phi_2+\hat{\theta}_2)\right)\dot{\hat{p}}_{11} - 4\,\tilde{\chi}_2^2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\hat{P}_{11} &= 0,\\ \ddot{\hat{P}}_{22} - \left(\dot{\hat{p}}_2(\tau)/\tilde{\chi}_2(\tau) - 2\,i\,\frac{d}{d\tau}(\phi_2+\hat{\theta}_2)\right)\dot{\hat{p}}_{22} - 4\,\tilde{\chi}_2^2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\hat{P}_{22} &= 0.\end{aligned} \tag{B10}$$

The differential equations (B10) do not admit an explicit solution in general. However, they can be integrated numerically when an explicit form of $\tilde{g}_2(\tau)$ and $\tilde{g}'(\tau)$ is given.

These differential equations have to be supplemented by initial conditions. We note that, since the left hand side of (B5) is the identity matrix for $\tau = 0$, we have that $\boldsymbol{P}(0) = 0$, which implies $\hat{P}_{11}(0) = \hat{P}_{22}(0) = \mathbb{1}$. In addition, taking the time derivative of both sides of (B5) and setting $t = 0$, it is easy to check that this implies $\frac{d}{d\tau}\hat{P}_{11}|_{\tau=0} = \frac{d}{d\tau}\hat{P}_{22}|_{\tau=0} = 0$.

Given that the differential equations are valid only when $\tilde{g}_2(\tau) \neq 0$, it is convenient to introduce the functions $\hat{p}_{11}(y)$ and $\hat{p}_{22}(y)$ defined as

$$\begin{aligned}\hat{P}_{11}(\tau) = \hat{p}_{11}(y) &:= \hat{p}_{11}\left(2\int_0^\tau d\tau'\tilde{\chi}_2(\tau')\right),\\ \hat{P}_{22}(\tau) = \hat{p}_{22}(y) &:= \hat{p}_{22}\left(2\int_0^\tau d\tau'\tilde{\chi}_2(\tau')\right),\end{aligned} \tag{B11}$$

where we have introduced $y(\tau) := 2\int_0^\tau d\tau'\tilde{\chi}_2(\tau')$. In terms of these functions, we have that the differential equations (B10) read

$$\begin{aligned}\tilde{\chi}_2(\tau)\,\ddot{\hat{p}}_{11} - i\,\frac{d}{d\tau}(\phi_2+\hat{\theta}_2)\,\dot{\hat{p}}_{11} - \tilde{\chi}_2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\hat{p}_{11} = 0,\\ \tilde{\chi}_2(\tau)\,\ddot{\hat{p}}_{22} + i\,\frac{d}{d\tau}(\phi_2+\hat{\theta}_2)\,\dot{\hat{p}}_{22} - \tilde{\chi}_2(\tau)\,(\hat{a}^\dagger\hat{a})^2\,\hat{p}_{22} = 0,\end{aligned} \tag{B12}$$

Published under license by AIP Publishing

where the derivative stands for derivative with respect to the variable y.

This has allowed us to find an expression for $\boldsymbol{S}_{\text{sq}}(\tau) = \boldsymbol{P} - i\, a^\dagger \hat{a} \int_0^\tau d\tau'\, \boldsymbol{K}\, \boldsymbol{P}$, which reads

$$\boldsymbol{S}_{\text{sq}}(\tau) = \begin{pmatrix} \hat{p}_{11} & -2\, i\, \hat{a}^\dagger \hat{a} \displaystyle\int_0^\tau d\tau'\, \tilde{\chi}_2(\tau')\, e^{2\, i\, (\phi_2 + \hat{\theta}_2)}\, \hat{p}_{22} \\ 2\, i\, \hat{a}^\dagger \hat{a} \displaystyle\int_0^\tau d\tau'\, \tilde{\chi}_2(\tau')\, e^{-2\, i\, (\phi_2 + \hat{\theta}_2)}\, \hat{p}_{11} & \hat{p}_{22} \end{pmatrix}. \tag{B13}$$

Given that we know that, in general, one has

$$\boldsymbol{S}_{\text{sq}}(\tau) = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\beta}^\dagger & \hat{\alpha}^\dagger \end{pmatrix}, \tag{B14}$$

this immediately allows us to identify the Bogoliubov coefficients as

$$\begin{aligned} \hat{\alpha} &= \hat{p}_{11}, \\ \hat{\beta} &= -2\, i\, \hat{a}^\dagger \hat{a} \int_0^\tau d\tau'\, \tilde{\chi}_2(\tau')\, e^{2\, i\, (\phi_2 + \hat{\theta}_2)}\, \hat{p}_{11}^\dagger, \end{aligned} \tag{B15}$$

with the auxiliary consistency condition $\hat{p}_{22} = \hat{p}_{11}^\dagger$. This condition also follows from (B10).

3. Useful expressions

Here, we present a list of useful expressions that are too cumbersome to appear in the main text but include key steps for the obtainment of the final results.

We start by presenting the expressions for $\Re\big[\tilde{g}_1(\tau)\, \hat{E}_2\, (\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\big]$ and $\Im\big[\tilde{g}_1(\tau)\, \hat{E}_2\, (\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\big]$ for the case of constant couplings $\tilde{\chi}_2$ and $\tilde{g}_2'$, which read

$$\begin{aligned} \Re\Big[\tilde{g}_1(\tau)\, \hat{E}_2\, (\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\Big] &= \tilde{g}_1^{(+)} \left[\cos\big(\hat{\Lambda}\, \tau\big) + 2\, \tilde{\chi}_2\, \frac{\hat{a}^\dagger \hat{a}}{\hat{\Lambda}}\, \sin(2\, \phi_2)\, \sin\big(\hat{\Lambda}\, \tau\big)\right] - \tilde{g}_1^{(-)} \left[\frac{\hat{K}}{\hat{\Lambda}} + 2\, \tilde{\chi}_2\, \frac{\hat{a}^\dagger \hat{a}}{\hat{\Lambda}}\, \cos(2\, \phi_2)\right] \sin\big(\hat{\Lambda}\, \tau\big), \\ \Im\Big[\tilde{g}_1(\tau)\, \hat{E}_2\, (\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\Big] &= \tilde{g}_1^{(+)} \left[\frac{\hat{K}}{\hat{\Lambda}} - 2\, \tilde{\chi}_2\, \frac{\hat{a}^\dagger \hat{a}}{\hat{\Lambda}}\, \cos(2\, \phi_2)\right] \sin\big(\hat{\Lambda}\, \tau\big) + \tilde{g}_1^{(-)} \left[\cos\big(\hat{\Lambda}\, \tau\big) - 2\, \tilde{\chi}_2\, \frac{\hat{a}^\dagger \hat{a}}{\hat{\Lambda}}\, \sin(2\, \phi_2)\, \sin\big(\hat{\Lambda}\, \tau\big)\right]. \end{aligned} \tag{B16}$$

This allows us to find

$$\begin{aligned} \hat{U}(t) := {}& e^{-i \int_0^\tau d\tau' \tilde{\omega}_c(\tau') \hat{a}^\dagger \hat{a} + i\, \hat{F}^{(2)}\, (\hat{a}^\dagger \hat{a})^2}\, e^{-i\, \hat{\theta}_2\, \hat{b}^\dagger \hat{b}}\, \hat{U}_{\text{sq}}\, e^{-i\, \hat{a}^\dagger \hat{a} \int_0^\tau d\tau'\, \tilde{g}_1^{(+)}(\tau) \left[\cos(\hat{\Lambda}\, \tau') + 2\, \tilde{\chi}_2\, \frac{a^\dagger \hat{a}}{\hat{\Lambda}}\, \sin(2\, \phi_2)\, \sin(\hat{\Lambda}\, \tau')\right] \hat{B}_+} \\ & \times e^{i\, \hat{a}^\dagger \hat{a} \left[\frac{\hat{K}}{\hat{\Lambda}} + 2\, \tilde{\chi}_2\, \frac{a^\dagger \hat{a}}{\hat{\Lambda}}\, \cos(2\, \phi_2)\right] \int_0^\tau d\tau'\, \tilde{g}_1^{(-)}(\tau')\, \sin(\hat{\Lambda}\, \tau')\, \hat{B}_+}\, e^{i\, \hat{a}^\dagger \hat{a} \left[\frac{\hat{K}}{\hat{\Lambda}} - 2\, \tilde{\chi}_2\, \frac{a^\dagger \hat{a}}{\hat{\Lambda}}\, \cos(2\, \phi_2)\right] \int_0^\tau d\tau'\, \tilde{g}_1^{(+)}(\tau')\, \sin(\hat{\Lambda}\, \tau')\, \hat{B}_-} \\ & \times e^{i\, \hat{a}^\dagger \hat{a} \int_0^\tau d\tau'\, \tilde{g}_1^{(-)}(\tau') \left[\cos(\hat{\Lambda}\, \tau') - 2\, \tilde{\chi}_2\, \frac{a^\dagger \hat{a}}{\hat{\Lambda}}\, \sin(2\, \phi_2)\, \sin(\hat{\Lambda}\, \tau')\right] \hat{B}_-}. \end{aligned} \tag{B17}$$

APPENDIX C: PROPERTIES OF THE FINAL STATE

The final state $\hat{\rho}(\tau)$ can be partially computed analytically, given the expressions for our initial state (6) and the time-evolution operator (12), and reads

$$\hat{\rho}(\tau) = e^{-|\mu|^2} \sum_{n,m,p} \frac{\tanh^{2p} r_T}{\cosh^2 r_T}\, \frac{\mu^n\, \mu^{*m}}{\sqrt{n!}\sqrt{m!}}\, \hat{U}(t) |n\rangle |p\rangle \langle m| \langle p| \hat{U}^\dagger(t), \tag{C1}$$

which can be made slightly more explicit as

$$\begin{aligned} \hat{\rho}(\tau) = {}& e^{-|\mu|^2} \sum_{n,m,p} \frac{\tanh^{2p} r_T}{\cosh^2 r_T}\, \frac{\mu^n\, \mu^{*m}}{\sqrt{n!}\sqrt{m!}}\, e^{-i \int_0^\tau d\tau' \tilde{\omega}_c(\tau')(n-m) + i\, (\hat{F}_n^{(2)}\, n^2 - \hat{F}_m^{(2)}\, m^2)} \\ & \times e^{-i\, \hat{\theta}_{2,n}\, \hat{b}^\dagger \hat{b}}\, \hat{U}_{\text{sq},n}\, e^{-i \int_0^\tau d\tau'\, n\, \Re\left[\tilde{g}_1(\tau')\, \hat{E}_2\, (\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\right]_n \hat{B}_+}\, e^{i \int_0^\tau d\tau'\, n\, \Im\left[\tilde{g}_1(\tau')\, \hat{E}_2\, (\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\right]_n \hat{B}_-} \\ & |n\rangle |p\rangle \langle m| \langle p| e^{-i \int_0^\tau d\tau'\, m\, \Im\left[\tilde{g}_1(\tau')\, \hat{E}_2\, (\hat{\alpha}^\dagger - \hat{\beta}^\dagger)\right]_m \hat{B}_-}\, e^{i \int_0^\tau d\tau'\, m\, \Re\left[\tilde{g}_1(\tau')\, \hat{E}_2\, (\hat{\alpha}^\dagger + \hat{\beta}^\dagger)\right]_m \hat{B}_+}\, \hat{U}_{\text{sq},m}^\dagger\, e^{i\, \hat{\theta}_{2,m}\, \hat{b}^\dagger \hat{b}}. \end{aligned} \tag{C2}$$

In expression (C2), the subscripts n and m mean that the respective quantities are evaluated for $\hat{a}^\dagger \hat{a} \to n, m$.

Published under license by AIP Publishing

1. Reduced final state of the mechanical oscillator

We can compute the final reduced state $\hat{\rho}_{\mathrm{m}}(\tau)$ of the mechanical resonator. We can obtain it as $\hat{\rho}_{\mathrm{m}}(\tau) := \mathrm{Tr}_{\mathrm{c}}(\hat{\rho}(\tau))$, which we can compute using (C2). We find

$$\hat{\rho}_{\mathrm{m}}(\tau) = e^{-|\mu|^2} \sum_{n,p} \frac{\tanh^{2p} r_T}{\cosh^2 r_T} \frac{|\mu|^{2n}}{n!} e^{-i\,\hat{\theta}_{2,n}\,\hat{b}^\dagger \hat{b}}\, \hat{U}_{\mathrm{sq},n}\, e^{-i \int_0^\tau d\tau'\, n\, \Re\left[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger+\hat{\beta}^\dagger)\right]_n \hat{B}_+}\, e^{i \int_0^\tau d\tau'\, n\, \Im\left[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger-\hat{\beta}^\dagger)\right]_n \hat{B}_-}$$
$$|p\rangle\langle p| e^{-i \int_0^\tau d\tau'\, n\, \Im\left[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger-\hat{\beta}^\dagger)\right]_n \hat{B}_-}\, e^{i \int_0^\tau d\tau'\, n\, \Re\left[\tilde{g}_1(\tau')\,\hat{E}_2\,(\hat{\alpha}^\dagger+\hat{\beta}^\dagger)\right]_n \hat{B}_+}\, \hat{U}^\dagger_{\mathrm{sq},n}\, e^{i\,\hat{\theta}_{2,n}\,\hat{b}^\dagger \hat{b}}. \tag{C3}$$

Note that here, again, the subscript n means that we need to replace $n \to \hat{a}^\dagger \hat{a}$.

APPENDIX D: MIXEDNESS IN THE FINAL REDUCED STATE OF THE MECHANICAL OSCILLATOR

Here, we compute the mixedness of the reduced state of the mechanical oscillator. Given that any pure state $\hat{\rho}$ has the property that $\hat{\rho}^2 = \hat{\rho}$, it follows that the mixedness of a state can be quantified by the *linear entropy* S_N defined as $S_N(\hat{\rho}) := 1 - \mathrm{Tr}(\hat{\rho}^2)$.

In general, although the expression (C3) gives us some insight into the reduced state of the mechanical mode, it is difficult to obtain the mixedness for arbitrary parameters of the Hamiltonian. We therefore proceed to specialize to an interesting, yet general enough, scenario below.

1. Mixedness of the reduced state of the mechanical oscillator: No cross-Kerr squeezing

In the case where $\tilde{g}_2 = 0$ we have $\hat{U}_{\mathrm{sq},n} = \mathbb{1}$, $\hat{\alpha} = \mathbb{1}$, and $\hat{\beta} = 0$. Therefore, with some algebra, we can show that expression (C3) simplifies to

$$\hat{\rho}_{\mathrm{m}}(\tau) = e^{-|\mu|^2} \sum_{n,p} \frac{\tanh^{2p} r_T}{\cosh^2 r_T} \frac{|\mu|^{2n}}{n!} e^{-i\,\hat{\theta}_2\,\hat{b}^\dagger \hat{b}}\, e^{-i \int_0^\tau d\tau'\, n\left[\tilde{g}_1^*(\tau')\, e^{-i\,\hat{\theta}_{2,n}}\,\hat{b}^\dagger + \tilde{g}_1(\tau')\, e^{i\,\hat{\theta}_{2,n}}\,\hat{b}\right]} |p\rangle\langle p|\, e^{i \int_0^\tau d\tau'\, n\left[\tilde{g}_1^*(\tau')\, e^{-i\,\hat{\theta}_{2,n}}\,\hat{b}^\dagger + \tilde{g}_1(\tau')\, e^{i\,\hat{\theta}_{2,n}}\,\hat{b}\right]}\, e^{i\,\hat{\theta}_2\,\hat{b}^\dagger \hat{b}}. \tag{D1}$$

Therefore, we can easily show that the linear entropy $S_N(\hat{\rho}_{\mathrm{m}}(\tau))$ associated with this state reads

$$S_N(\hat{\rho}_{\mathrm{m}}(\tau)) = 1 - e^{-2|\mu|^2} \sum_{n,n',p,p'} \frac{\tanh^{2p+2p'} r_T}{\cosh^4 r_T} \frac{|\mu|^{2n}}{n!} \frac{|\mu|^{2n'}}{n'!} \left|\langle p|\hat{D}_{nn'}(\tau)|p'\rangle\right|^2, \tag{D2}$$

where we have introduced

$$\hat{D}_{nn'}(\tau) := e^{i \int_0^\tau d\tau'\, n\left[\tilde{g}_1^*(\tau')\, e^{-i\,\hat{\theta}_{2,n}}\,\hat{b}^\dagger + \tilde{g}_1(\tau')\, e^{i\,\hat{\theta}_{2,n}}\,\hat{b}\right]}\, e^{-i \int_0^\tau d\tau'\, n'\left[\tilde{g}_1^*(\tau')\, e^{-i\,\hat{\theta}_{2,n'}}\,\hat{b}^\dagger + \tilde{g}_1(\tau')\, e^{i\,\hat{\theta}_{2,n'}}\,\hat{b}\right]}$$
$$= e^{i\,\psi_{nn'}}\, e^{i\,\Delta_{nn'}\,\hat{b}}\, e^{i\,\Delta_{nn'}\,\hat{b}^\dagger} \tag{D3}$$

and defined $\Delta_{nn'} := \int_0^\tau d\tau'\, \tilde{g}_1(\tau') \left(n\, e^{i\,\tau' + 2\,i\,n \int_0^{\tau'} d\tau''\, \tilde{g}_2'(\tau'')} - n'\, e^{i\,\tau' + 2\,i\,n' \int_0^{\tau'} d\tau''\, \tilde{g}_2'(\tau'')} \right)$. Note that the exact form of $e^{i\,\psi_{nn'}}$ does not matter since it cancels out in (D2).

It is also easy to check that $S_N(\hat{\rho}_{\mathrm{m}}(\tau))$ can also be written as

$$S_N(\hat{\rho}_{\mathrm{m}}(\tau)) = 1 - 2\,e^{-2|\mu|^2} \sum_{n,n',p=0} \sum_{k=1} \frac{\tanh^{4p+2k} r_T}{\cosh^4 r_T} \frac{|\mu|^{2n}}{n!} \frac{|\mu|^{2n'}}{n'!} \left|\langle p|\hat{D}_{nn'}(\tau)|p+k\rangle\right|^2 - e^{-2|\mu|^2} \sum_{n,n',p=0} \frac{\tanh^{4p} r_T}{\cosh^4 r_T} \frac{|\mu|^{2n}}{n!} \frac{|\mu|^{2n'}}{n'!} \left|\langle p|\hat{D}_{nn'}(\tau)|p\rangle\right|^2. \tag{D4}$$

Now, we can follow the same procedure introduced in Ref. 6 and compute (D4) explicitly. We find

$$S_N(\hat{\rho}_{\mathrm{m}}(\tau)) = 1 - e^{-2|\mu|^2} \sum_{n,n'} \frac{|\mu|^{2n+2n'}}{n!\,n'!} \frac{\exp\left[-\frac{1}{\cosh r_T} |\Delta_{nn'}|^2\right]}{\cosh r_T}. \tag{D5}$$

We note that, as expected, the result (D5) reduces to the existing one in the literature for $\tilde{g}_2'(\tau) = 0$, see Ref. 6.

REFERENCES

[1] J. Wei and E. Norman, J. Math. Phys. **4**, 575 (1963).
[2] R. R. Puri, *Mathematical Methods of Quantum Optics* (Springer Science & Business Media, 2001), Vol. 79.
[3] D. E. Bruschi, A. R. Lee, and I. Fuentes, J. Phys. A: Math. Theor. **46**, 165303 (2013).
[4] G. Adesso, S. Ragy, and A. R. Lee, Open Syst. Inf. Dyn. **21**, 1440001 (2014).
[5] D. E. Bruschi and A. Xuereb, New J. Phys. **20**, 065004 (2018).
[6] D. E. Bruschi, J. Math. Phys. **60**, 062105 (2019).
[7] S. Qvarfort, A. Serafini, A. Xuereb, D. Rätzel, and D. E. Bruschi, New J. Phys. **21**, 055004 (2019).
[8] E. Buks, C. R. Phys. **13**, 454 (2012).
[9] M. R. Vanner, I. Pikovski, G. D. Cole, M. S. Kim, Č. Brukner, K. Hammerer, G. J. Milburn, and M. Aspelmeyer, Proc. Natl. Acad. Sci. U. S. A. **108**, 16182 (2011).
[10] H. Ritsch, P. Domokos, F. Brennecke, and T. Esslinger, Rev. Mod. Phys. **85**, 553 (2013).
[11] E.-j. Kim, J. R. Johansson, and F. Nori, Phys. Rev. A **91**, 033835 (2015).
[12] D. Rätzel, F. Schneiter, D. Braun, T. Bravo, R. Howl, M. P. E. Lock, and I. Fuentes, New J. Phys. **20**, 053046 (2018).
[13] J. Schmöle, M. Dragosits, H. Hepach, and M. Aspelmeyer, Classical Quantum Gravity **33**, 125031 (2016).
[14] P. Barberis-Blostein and I. Fuentes-Schuller, Phys. Rev. A **78**, 013641 (2008).
[15] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[16] E. G. Brown, E. Martín-Martínez, N. C. Menicucci, and R. B. Mann, Phys. Rev. D **87**, 084062 (2013).
[17] C. Moore and D. E. Bruschi, "Tuneable interacting bosons for relativistic and quantum information processing," arXiv:1601.01919 (2016).
[18] D. E. Bruschi, C. Sabín, and G. S. Paraoanu, Phys. Rev. A **95**, 062324 (2017).
[19] P. Lähteenmäki, G. S. Paraoanu, J. Hassel, and P. J. Hakonen, Nat. Commun. **7**, 12548 (2016).
[20] D. Dong and I. R. Petersen, IET Control Theory Appl. **4**, 2651 (2010).
[21] C. F. Ockeloen-Korppi, E. Damskägg, G. S. Paraoanu, F. Massel, and M. A. Sillanpää, Phys. Rev. Lett. **121**, 243601 (2018).
[22] M. Paternostro, D. Vitali, S. Gigan, M. S. Kim, C. Brukner, J. Eisert, and M. Aspelmeyer, Phys. Rev. Lett. **99**, 250401 (2007).
[23] J. Williamson, Am. J. Math. **58**, 141 (1936).

Published under license by AIP Publishing