Chapter 1
Computing elastic interior transmission eigenvalues

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1.1 Introduction

Non-destructive testing is an important tool to check whether a given object is homogeneous or not without destroying it. Interior transmission eigenvalues (ITE) may have the potential to serve as an indicator whether an object is homogeneous or not due to a monotonicity result. If the object is not homogeneous, they might indicate where and how large the inhomogeneity is. Hence, they can be seen as a "fingerprint" of a given object. Therefore, it is of great interest to numerically calculate them for arbitrary domains to high accuracy.

They also play an important role in the theory for scattering problems. Precisely, algorithms such as the (general) linear sampling method or the factorization method to reconstruct the scattering object from the scattered field are not theoretically justified for such eigenvalues. Usually, time-harmonic acoustic, electromagnetic, or elastic scattering problems are considered. Recent work is now focusing on the latter one as we do here, too.

Unfortunately, the resulting system of partial differential equations, containing two Navier equations, are coupled by transmission conditions and lead therefore to a non-self-adjoint and non-elliptic problem. However, one can cope with this problem. Existing methods like the inside-outside duality method [Pe16] do not report numerical results, the method of fundamental solutions only works well for small perturbations of a circle [KlPi20], and variants of the finite element method

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only work well for polygonal domains [ChLiWa20, JiLiSu18, JiLiSu20, XiJi18, XiJiGe18, XiJiZh21, YaHaBi20, YaEtAl20].

An alternative is to use the boundary element method which works very well for domains with smooth boundaries to obtain numerical results to high accuracy. However, it can only be used for constant coefficients and the fundamental solution needs to be known. Luckily this is the case for the Navier equation, but one needs to solve a non-linear eigenvalue problem which can be done with Beyn's algorithm [Be12] as done for the acoustic transmission problem [Kl13]. A first attempt has been made in [We18], but certain integral operators were too complicated to be approximated. An improvement is given in [Zi21] fully avoiding this integral operator by using a difference of Dirichlet-to-Neumann maps which has been successfully applied to the acoustic transmission problem in [CaKr17]. However, the numerical approximation of the singular integrals to high accuracy is complicated. Here, we use an approach which fully avoids the numerical calculation of singular integrals.

The existence of a countable number of real ITEs is known [BeCaGu13], but the existence of complex ITEs is still open, but with our approach we are able to give numerical results indicating that they do exist.

The chapter is organized as follows: First, we present the elastic interior transmission problem. Next, we illustrate how to solve it with the boundary element method using a difference of Dirichlet-to-Neumann maps. Then, the resulting integral equation is approximated by the boundary element collocation method and the emerging non-linear eigenvalue problem is solved with Beyn's algorithm. Numerical results are given to show the correct approximations for two test cases. Finally, numerical results are reported for a variety of domains and compared with existing results. A short summary and an outlook are given at the end.

1.2 Elastic transmission eigenvalue problem

Let $D \subset \mathbb{R}^2$ be a bounded open domain that is simply connected. Its boundary ∂D is given parametrically by $\mathbf{p}(\theta)$ with $\theta \in [0, 2\pi]$. We assume that ∂D is a simple, closed curve with finite length satisfying $\mathbf{p}(0) = \mathbf{p}(2\pi)$, $\mathbf{p} \in C^2([0, 2\pi])$, and $\mathbf{p}'(\theta) \neq \mathbf{0}$ for all $\theta \in [0, 2\pi]$.

Time-harmonic elastic scattering with frequency ω can be described by the Navier equation

$$\mu\, \Delta \mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} + \omega^2 \rho\, \mathbf{u} = \mathbf{0} \quad \text{in } D \subset \mathbb{R}^2\,, \tag{1.1}$$

where $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))^\top$ is the displacement field at the point $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$. Here, the parameter $\rho > 0$ is the mass density of the medium and assumed to be constant. The parameters λ and μ are the Lamé parameters and describe the elastic material. They satisfy the conditions $\mu > 0$ and $2\mu + \lambda > 0$ ([Mc00, p. 297 ff.]).

Assume now that D with mass density ρ_1 is contained in a medium with mass density ρ_0 with $\rho_1 > \rho_0$. Is there an incident field satisfying the Navier equation that does not scatter? This leads to the elastic interior transmission problem: Find ω^2 and a non-trivial solution $(\mathbf{u},\mathbf{v})$ such that

$$\mu\,\Delta\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\omega^2\rho_0\,\mathbf{u}\;=\mathbf{0}\qquad\text{in } D\,, \tag{1.2}$$

$$\mu\,\Delta\mathbf{v}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}+\omega^2\rho_1\,\mathbf{v}\;=\mathbf{0}\qquad\text{in } D\,, \tag{1.3}$$

$$\mathbf{u}=\mathbf{v}\qquad\text{on } \partial D\,, \tag{1.4}$$

$$\mathbf{T}(\mathbf{u})=\mathbf{T}(\mathbf{v})\ \text{on } \partial D\,. \tag{1.5}$$

is satisfied, where

$$\mathbf{T}(\mathbf{f})=\lambda\,\mathrm{div}(\mathbf{f})\,\nu+2\,\mu\,(\nu^\top\,\mathrm{grad})\,\mathbf{f}+\mu\,\mathrm{div}(\mathrm{Q}\mathbf{f})\,\mathrm{Q}\,\nu$$

with the normalized vector $\nu=(\nu_1,\nu_2)^\top$ on ∂D pointing into the exterior of D and the matrix

$$\mathrm{Q}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Then, the parameter ω is an elastic interior transmission eigenvalue (EITE). The existence of real EITEs is known [BeCaGu13], however the existence of complex EITEs is still open.

We will use boundary integral equations to solve the problem at hand. The matrix-valued fundamental solution is given by

$$\begin{aligned}\mathrm{K}_\omega(\mathbf{x},\mathbf{y})&=\frac{\mathrm{i}}{4\mu}H_0^{(1)}(k_s\|\mathbf{x}-\mathbf{y}\|)\mathrm{I}_2\\&\quad+\frac{\mathrm{i}}{4\omega^2}\,\mathrm{grad}_{\mathbf{x}}\,\mathrm{grad}_{\mathbf{x}}^\top\left[H_0^{(1)}(k_s\|\mathbf{x}-\mathbf{y}\|)-H_0^{(1)}(k_p\|\mathbf{x}-\mathbf{y}\|)\right]\in\mathbb{C}^{2x2}\,,\end{aligned}$$

where $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$ with $\mathbf{x}\neq\mathbf{y}$, $\|\cdot\|$ denotes the Euclidean norm, and I_2 is the 2×2 identity matrix. The function $H_0^{(1)}$ is the Hankel function of the first kind of order 0. The parameters k_p and k_s are the wave numbers of the shear and the pressure wave, respectively. They are given by

$$k_s^2=\frac{\omega^2}{\mu}\qquad\text{and}\qquad k_p^2=\frac{\omega^2}{\lambda+2\mu}\,.$$

The elastic single layer operator defined by

$$\mathbf{u}(\mathbf{x})=(\mathrm{SL}_\omega\mathbf{g})\,(\mathbf{x})=\int_{\partial D}\mathrm{K}_\omega(\mathbf{x},\mathbf{y})\mathbf{g}(\mathbf{y})\,\mathrm{d}s(\mathbf{y})\,,\quad \mathbf{x}\in D$$

as well as the elastic double layer operator defined by

$$\mathbf{u}(\mathbf{x})=(\mathrm{DL}_\omega\mathbf{h})\,(\mathbf{x})=\int_{\partial D}\left[\mathbf{T}_{\mathbf{y}}\left(\mathrm{K}_\omega(\mathbf{x},\mathbf{y})\right)\right]^\top\mathbf{h}(\mathbf{y})\,\mathrm{d}s(\mathbf{y})\,,\quad \mathbf{x}\in D$$

with unknown functions $\mathbf{g}$, $\mathbf{h}$ solve the Navier equation (1.1). Note that the traction of a matrix is applied to each column. The unknown functions $\mathbf{g}$, $\mathbf{h}$ are then determined by letting the point $\mathbf{x} \in D$ approach the boundary and using the given boundary condition incorporating the jump conditions of the elastic boundary layer operators defined by

$$\begin{aligned}
(\mathrm{S}_\omega \mathbf{g})(\mathbf{x}) &= \int_{\partial D} \mathrm{K}_\omega(\mathbf{x},\mathbf{y})\mathbf{g}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \partial D,\\
\left(\mathrm{D}_\omega^\top \mathbf{g}\right)(\mathbf{x}) &= \int_{\partial D} \mathbf{T}_{\mathbf{y}}\left(\mathrm{K}_\omega(\mathbf{x},\mathbf{y})\right)\mathbf{g}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \partial D,\\
(\mathrm{D}_\omega \mathbf{h})(\mathbf{x}) &= \int_{\partial D} \left[\mathbf{T}_{\mathbf{y}}\left(\mathrm{K}_\omega(\mathbf{x},\mathbf{y})\right)\right]^\top \mathbf{h}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \partial D.
\end{aligned}$$

The first operator is the elastic boundary single layer operator, the second operator is the traction of the elastic boundary single layer operator, and the third operator is the elastic boundary double layer operator. To solve (1.2)–(1.5), we use the idea given in [CaKr17]. The following ansatz

$$\mathbf{u} = \mathrm{SL}_{\omega\sqrt{\rho_0}}\mathbf{g} \quad \text{and} \quad \mathbf{v} = \mathrm{SL}_{\omega\sqrt{\rho_1}}\mathbf{h}$$

solves (1.2) and (1.3) in D. The functions $\mathbf{g}$ and $\mathbf{h}$ are unknown. Letting the point approach the boundary yields

$$\mathbf{u} = \mathrm{S}_{\omega\sqrt{\rho_0}}\mathbf{g} \quad \text{and} \quad \mathbf{v} = \mathrm{S}_{\omega\sqrt{\rho_1}}\mathbf{h} \quad \text{on } \partial D.$$

Taking the traction along with the jump conditions yields

$$\mathbf{T}(\mathbf{u}) = \left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_0}}\right)\mathbf{g} \quad \text{and} \quad \mathbf{T}(\mathbf{v}) = \left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_1}}\right)\mathbf{h} \quad \text{on } \partial D,$$

where I denotes the identity operator. Combining the last two equations gives

$$\mathbf{T}(\mathbf{u}) = \left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_0}}\right)\mathrm{S}^{-1}_{\omega\sqrt{\rho_0}}\mathbf{u} \text{ and} \tag{1.6}$$

$$\mathbf{T}(\mathbf{v}) = \left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_1}}\right)\mathrm{S}^{-1}_{\omega\sqrt{\rho_1}}\mathbf{v} \quad \text{on } \partial D. \tag{1.7}$$

where we assume that $\omega^2\rho_0$ and $\omega^2\rho_1$ are not eigenvalues of the operator $\Delta^* := \mu\,\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u}$ with boundary condition $\mathbf{u} = \mathbf{0}$. Because of the boundary condition (1.4), we can replace $\mathbf{v}$ by $\mathbf{u}$ in (1.7). Next, we take the difference of (1.6) and (1.7) and apply the boundary condition (1.5) yielding

$$\underbrace{\left[\left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_0}}\right)\mathrm{S}^{-1}_{\omega\sqrt{\rho_0}} - \left(\frac{1}{2}\mathrm{I} + \mathrm{D}^\top_{\omega\sqrt{\rho_1}}\right)\mathrm{S}^{-1}_{\omega\sqrt{\rho_1}}\right]}_{=:\mathrm{N}(\omega)}\mathbf{u} = \mathbf{0} \quad \text{on } \partial D.$$

Then, the solution of the non-linear eigenvalue problem $\mathrm{N}(\omega)\mathbf{u}=\mathbf{0}$, $\mathbf{u}\neq\mathbf{0}$ will be a solution of (1.2)–(1.5). However, we will consider the transpose of this equation to avoid the use of the traction of the elastic single layer operator. Hence, we consider

$$\underbrace{\left[\mathrm{S}^{-1}_{\omega\sqrt{\rho_0}}\left(\frac{1}{2}\mathrm{I}+\mathrm{D}_{\omega\sqrt{\rho_0}}\right)-\mathrm{S}^{-1}_{\omega\sqrt{\rho_1}}\left(\frac{1}{2}\mathrm{I}+\mathrm{D}_{\omega\sqrt{\rho_1}}\right)\right]}_{=:\mathrm{M}(\omega)}\mathbf{u}=\mathbf{0}\quad \text{on } \partial D\,.$$

and need to solve the problem

$$\mathrm{M}(\omega)\mathbf{u}=\mathbf{0}\,,\quad \mathbf{u}\neq\mathbf{0}$$

assuming $\omega^2\rho_0$ and $\omega^2\rho_1$ are not eigenvalues of the operator Δ^* with boundary condition $\mathbf{u}=\mathbf{0}$.

1.3 The discretization of the operators $\frac{1}{2}\mathrm{I}+\mathrm{D}_\omega$ and S_ω

In this section, we illustrate how to solve a given boundary integral equation with the boundary element collocation method which we will also use later to approximate the operators S_ω and $\frac{1}{2}\mathrm{I}+\mathrm{D}_\omega$ for a given ω. As an illustrative example, we want to solve the problem $\Delta^*\mathbf{u}+\omega^2\mathbf{u}=0$ in $\mathbb{R}^2\backslash\overline{D}$ with the boundary conditions $\mathbf{u}=\mathbf{f}$, where $\mathbf{f}$ is a given function defined on the boundary. The frequency ω is given as well. Using the double layer ansatz $\mathbf{u}=\mathrm{DL}_\omega\mathbf{h}$ in $\mathbb{R}^2\backslash\overline{D}$ together with the jump condition yields the boundary integral equation of the second kind

$$\frac{1}{2}\mathbf{h}+\mathrm{D}_\omega\mathbf{h}=\mathbf{f}\,. \tag{1.8}$$

Now, we illustrate how to solve this equation numerically. First, we define for a given even n the equidistant angles $\theta_j=2\pi(j-1)/n$, $j=1,\ldots,n$. With this, we define the nodes $\mathbf{v}_j=\mathbf{p}(\theta_j)$. Next, we define the line segments $\Delta_i\subset\partial D$, where the i-th segment has the starting point $\mathbf{v}_{2i-1}$ and the end point $\mathbf{v}_{2i+1}$ and a point in between $\mathbf{v}_{2i}$, $i=1,\ldots,n/2$. Note that $\mathbf{v}_{n+1}=\mathbf{v}_1$ since ∂D is closed. Hence, the given boundary ∂D can be written as the union of all Δ_i. Therefore, equation (1.8) can be written as

$$\frac{1}{2}\mathbf{h}(\mathbf{x})+\sum_{i=1}^{n/2}\int_{\Delta_i}\left[\mathbf{T}_\mathbf{y}\left(\mathrm{K}_\omega(\mathbf{x},\mathbf{y})\right)\right]^\top\mathbf{h}(\mathbf{y})\,\mathrm{d}s(\mathbf{y})=\mathbf{f}(\mathbf{x})\,,\quad \mathbf{x}\in\partial D\,.$$

It can be shown that there exists a bijective map $\mathbf{m}_i:\sigma=[0,1]\to\Delta_i$ for each $i=1,\ldots,n/2$. Using a change of variables yields

$$\frac{1}{2}\mathbf{h}(\mathbf{x})+\sum_{i=1}^{n/2}\int_{\sigma}\left[\mathbf{T}_{\mathbf{m}_i(s)}\left(\mathrm{K}_\omega(\mathbf{x},\mathbf{m}_i(s))\right)\right]^\top\mathbf{h}(\mathbf{m}_i(s))J_i(s)\,\mathrm{d}s(s)=\mathbf{f}(\mathbf{x})\,,\quad \mathbf{x}\in\partial D\,,$$

where $J_i(s) = \|\partial_s \mathbf{m}_i(s)\|$ is the Jacobian. The map $\mathbf{m}_i$ is approximated by a quadratic interpolation polynomial $\widetilde{\mathbf{m}}_i(s) = \sum_{j=1}^3 \mathbf{m}_i(q_j)L_j(s)$ with the Lagrange basis function $L_1(s) = (1-s)(1-2s)$, $L_2(s) = 4s(1-s)$, and $L_3(s) = s(2s-1)$ and $q_1 = 0$, $q_2 = 1/2$, and $q_3 = 1$. Note that $\mathbf{m}_i(q_j)$ selects the corresponding nodes $\mathbf{v}_{2i-1}$, $\mathbf{v}_{2i}$, and $\mathbf{v}_{2i+1}$. We approximately obtain

$$\frac{1}{2}\mathbf{h}(\mathbf{x}) + \sum_{i=1}^{n/2} \int_\sigma \left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_\omega(\mathbf{x}, \widetilde{\mathbf{m}}_i(s))\right)\right]^\top \mathbf{h}(\widetilde{\mathbf{m}}_i(s))\widetilde{J}_i(s)\,\mathrm{d}s(s) \approx \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \partial D,$$

where $\widetilde{J}_i(s) = \|\partial_s \widetilde{\mathbf{m}}_i(s)\|$ is the Jacobian. We define for a given $0 < \alpha < 1/2$ the collocation nodes $\widetilde{\mathbf{v}}_{i,k} = \widetilde{\mathbf{m}}_i(\widetilde{q}_k)$ for $i = 1, \ldots, n/2$ and $k = 1, 2, 3$ with $\widetilde{q}_1 = \alpha$, $\widetilde{q}_2 = 1/2$, and $\widetilde{q}_3 = 1 - \alpha$. We now approximate each component of the unknown function $\mathbf{h}$ by a quadratic interpolation polynomial using the three nodes $\widetilde{q}_k$ and the three Lagrange basis functions

$$\widetilde{L}_1(s) = \frac{1-s-\alpha}{1-2\alpha}\frac{1-2s}{1-2\alpha}, \; \widetilde{L}_2(s) = 4\frac{s-\alpha}{1-2\alpha}\frac{1-s-\alpha}{1-2\alpha}, \; \widetilde{L}_3(s) = \frac{s-\alpha}{1-2\alpha}\frac{2s-1}{1-2\alpha}.$$

Precisely, we use

$$\mathbf{h}(\widetilde{\mathbf{m}}_i(s)) \approx \sum_{k=1}^3 \mathbf{h}(\widetilde{\mathbf{m}}_i(\widetilde{q}_k))\widetilde{L}_k(s) = \sum_{k=1}^3 \mathbf{h}(\widetilde{\mathbf{v}}_{i,k})\widetilde{L}_k(s)$$

and therefore, we obtain

$$\begin{aligned}\frac{1}{2}\mathbf{h}(\mathbf{x}) &+ \sum_{i=1}^{n/2}\sum_{k=1}^{3} \int_\sigma \left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_\omega(\mathbf{x}, \widetilde{\mathbf{m}}_i(s))\right)\right]^\top \widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s)\mathbf{h}(\widetilde{\mathbf{v}}_{i,k}) - \mathbf{f}(\mathbf{x}) \\ &\approx \mathbf{r}(x), \quad \mathbf{x} \in \partial D,\end{aligned}$$

with the residual $\mathbf{r}(x)$. We force the residual to be zero at the collocation nodes $\widetilde{\mathbf{v}}_{j,\ell}$, which leads to the linear system of size $3n \times 3n$

$$\frac{1}{2}\mathbf{h}(\widetilde{\mathbf{v}}_{j,\ell}) + \sum_{i=1}^{n/2}\sum_{k=1}^{3} {}^{\omega}A_{(i,k),(j,\ell)}\mathbf{h}(\widetilde{\mathbf{v}}_{i,k}) = \mathbf{f}(\widetilde{\mathbf{v}}_{j,\ell}) \tag{1.9}$$

with

$${}^{\omega}A_{(i,k),(j,\ell)} = \int_\sigma \left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_\omega(\widetilde{\mathbf{v}}_{j,\ell}, \widetilde{\mathbf{m}}_i(s))\right)\right]^\top \widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s) \in \mathbb{C}^{2\times 2}$$

since the $(i,k),(j,\ell)$-entry is a 2×2 matrix. All four elements of the 2×2 matrix are

$$\begin{aligned}{}^{\omega}A^{(1,1)}_{(i,k),(j,\ell)} &= \int_\sigma t^{(1,1)}_{i,j,\ell}(s)\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s), \; {}^{\omega}A^{(1,2)}_{(i,k),(j,\ell)} = \int_\sigma t^{(1,2)}_{i,j,\ell}(s)\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s) \\ {}^{\omega}A^{(2,1)}_{(i,k),(j,\ell)} &= \int_\sigma t^{(2,1)}_{i,j,\ell}(s)\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s), \; {}^{\omega}A^{(2,2)}_{(i,k),(j,\ell)} = \int_\sigma t^{(2,2)}_{i,j,\ell}(s)\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s)\end{aligned}$$

with

$$
\begin{aligned}
\mathbf{t}_{i,j,\ell}^{(1,1)}(s) =\ & \frac{c_1}{\|\mathbf{d}_{i,j,\ell}(s)\|}\left[(-\lambda-2\mu)\nu_1(y)d_{i,j,\ell}^{(1)}(s)-\mu\nu_2(y)d_{i,j,\ell}^{(2)}(s)\right] \\
&+ \frac{c_2}{\|\mathbf{d}_{i,j,\ell}(s)\|^3}\left[(-\lambda-2\mu)\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^3\right. \\
&- \left.\lambda\nu_1(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2-2\mu\nu_2(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)\right] \\
&+ \frac{c_3}{\|\mathbf{d}_{i,j,\ell}(s)\|^4}\left[(-\lambda-4\mu)\nu_1(y)d_{i,j,\ell}^{(1)}(s)-\mu\nu_2(y)d_{i,j,\ell}^{(2)}(s)\right. \\
&+ \left.4\mu\left(\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^3+\nu_2(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)\right)\right] \\
\mathbf{t}_{i,j,\ell}^{(1,2)}(s) =\ & \frac{c_1}{\|\mathbf{d}_{i,j,\ell}(s)\|}\left[-\lambda\nu_2(y)d_{i,j,\ell}^{(1)}(s)-\mu\nu_1(y)d_{i,j,\ell}^{(2)}(s)\right] \\
&+ \frac{c_2}{\|\mathbf{d}_{i,j,\ell}(s)\|^3}\left[(-\lambda-2\mu)\nu_2(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2\right. \\
&- \left.\lambda\nu_2(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^3-2\mu\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)\right] \\
&+ \frac{c_3}{\|\mathbf{d}_{i,j,\ell}(s)\|^4}\left[(-\lambda-2\mu)\nu_2(y)d_{i,j,\ell}^{(1)}(s)-\mu\nu_1(y)d_{i,j,\ell}^{(2)}(s)\right. \\
&+ \left.4\mu\left(\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)+\nu_2(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2\right)\right] \\
\mathbf{t}_{i,j,\ell}^{(2,1)}(s) =\ & \frac{c_1}{\|\mathbf{d}_{i,j,\ell}(s)\|}\left[-\lambda\nu_1(y)d_{i,j,\ell}^{(2)}(s)-\mu\nu_2(y)d_{i,j,\ell}^{(1)}(s)\right] \\
&+ \frac{c_2}{\|\mathbf{d}_{i,j,\ell}(s)\|^3}\left[(-\lambda-2\mu)\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)\right. \\
&- \left.\lambda\nu_1(y)\left(d_{i,j,\ell}^{(2)}(s)\right)^3-2\mu\nu_2(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2\right] \\
&+ \frac{c_3}{\|\mathbf{d}_{i,j,\ell}(s)\|^4}\left[(-\lambda-2\mu)\nu_1(y)d_{i,j,\ell}^{(2)}(s)-\mu\nu_2(y)d_{i,j,\ell}^{(1)}(s)\right. \\
&+ \left.4\mu\left(\nu_1(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)+\nu_2(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2\right)\right] \\
\mathbf{t}_{i,j,\ell}^{(2,2)}(s) =\ & \frac{c_1}{\|\mathbf{d}_{i,j,\ell}(s)\|}\left[(-\lambda-2\mu)\nu_2(y)d_{i,j,\ell}^{(2)}(s)-\mu\nu_1(y)d_{i,j,\ell}^{(1)}(s)\right] \\
&+ \frac{c_2}{\|\mathbf{d}_{i,j,\ell}(s)\|^3}\left[(-\lambda-2\mu)\nu_2(y)\left(d_{i,j,\ell}^{(2)}(s)\right)^3\right. \\
&- \left.\lambda\nu_2(y)\left(d_{i,j,\ell}^{(1)}(s)\right)^2 d_{i,j,\ell}^{(2)}(s)-2\mu\nu_1(y)d_{i,j,\ell}^{(1)}(s)\left(d_{i,j,\ell}^{(2)}(s)\right)^2\right] \\
&+ \frac{c_3}{\|\mathbf{d}_{i,j,\ell}(s)\|^4}\left[(-\lambda-4\mu)\nu_2(y)d_{i,j,\ell}^{(2)}(s)-\mu\nu_1(y)d_{i,j,\ell}^{(1)}(s)\right.
\end{aligned}
$$

$$+\ 4\mu\left(\nu_1(y)d^{(1)}_{i,j,\ell}(s)\left(d^{(2)}_{i,j,\ell}(s)\right)^2+\nu_2(y)\left(d^{(2)}_{i,j,\ell}(s)\right)^3\right)\Bigg]$$

where

$$\begin{aligned}
\mathbf{d}_{i,j,\ell}(s) &= \widetilde{\mathbf{v}}_{j,\ell}-\widetilde{\mathbf{m}}_i(s)\\
c_1 &= -\frac{\mathrm{i}k_s}{4\mu}H^{(1)}_1(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)\\
&-\ \frac{\mathrm{i}}{4\omega^2\,\|\mathbf{d}_{i,j,\ell}(s)\|}\left[2\frac{k_pH^{(1)}_1(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_sH^{(1)}_1(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)}{\|\mathbf{d}_{i,j,\ell}(s)\|}\right.\\
&-\ \left.k_p^2H^{(1)}_0(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)\ +k_s^2H^{(1)}_0(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)\right],\\
c_2 &= \frac{\mathrm{i}}{2\omega^2\,\|\mathbf{d}_{i,j,\ell}(s)\|}\ \left[k_s^2H^{(1)}_0(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_p^2H^{(1)}_0(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)\right.\\
&+\ \left.2\frac{k_pH^{(1)}_1(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_sH^{(1)}_1(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)}{\|\mathbf{d}_{i,j,\ell}(s)\|}\right]\\
&+\ \frac{\mathrm{i}}{4\omega^2}\ \left[k_s^3H^{(1)}_1(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_p^3H^{(1)}_1(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)\right],\\
c_3 &= \frac{\mathrm{i}}{2\omega^2\,\|\mathbf{d}_{i,j,\ell}(s)\|}\ \left[k_sH^{(1)}_1(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_pH^{(1)}_1(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)\right]\\
&+\ \frac{\mathrm{i}}{4\omega^2}\ \left[k_p^2H^{(1)}_0(k_p\,\|\mathbf{d}_{i,j,\ell}(s)\|)-k_s^2H^{(1)}_0(k_s\,\|\mathbf{d}_{i,j,\ell}(s)\|)\right]
\end{aligned}$$

The four integrals in $A_{(i,k),(j,\ell)}$ have to be evaluated numerically which is done with an automatic integration routine using adaptive quadrature (refer to the software package QUADPACK). However, when $(i,k)=(j,\ell)$ a singularity is present. In this case, we use a singularity subtraction of the form

$$\begin{aligned}
{}^{\omega}A_{(i,k),(i,k)} &= \int_\sigma\left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_\omega(\widetilde{\mathbf{v}}_{i,k},\widetilde{\mathbf{m}}_i(s))-\mathrm{K}_0(\widetilde{\mathbf{v}}_{i,k},\widetilde{\mathbf{m}}_i(s))\right)\right]^\top\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s)\\
&+\ \underbrace{\int_\sigma\left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_0(\widetilde{\mathbf{v}}_{i,k},\widetilde{\mathbf{m}}_i(s))\right)\right]^\top\widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s)}_{{}^{0}A_{(i,k),(i,k)}}\\
&=:\ \mathrm{int}^{\mathrm{smooth}}_{i,k}+\mathrm{int}^{\mathrm{singular}}_{i,k}.
\end{aligned}$$

The integrand of $\mathrm{int}^{\mathrm{smooth}}_{i,k}$ is smooth and converges to the 2×2 zero matrix, say Z_2. Therefore, we directly set $\mathrm{int}^{\mathrm{smooth}}_{i,k}$ equal to Z_2. Next, we consider $\mathrm{int}^{\mathrm{singular}}_{i,k}$. We use the fact that for $\phi=1$ we have $\mathrm{D}_0\phi(x)=-\frac{1}{2}I_2$ for all $x\in\partial D$. Hence, we approximately have

$$\sum_{i=1}^{n/2}\sum_{k=1}^{3} {}^0A_{(i,k),(j,\ell)} \approx -\frac{1}{2}I_2 \qquad \forall\,(j,\ell) \tag{1.10}$$

and therefore we can find the diagonal matrix entry ${}^0A_{(i,k),(i,k)} = \mathrm{int}_{i,k}^{\text{singular}}$ by enforcing (1.10) to be exact. Hence, we never have to integrate over a singularity, but we need to additionally compute the 2×2 matrices ${}^0A_{(i,k),(j,\ell)}$ for all $(i,k)\neq(j,\ell)$. For a given $\mathbf{f}$ and ω, the linear system (1.9) is solved directly for $\mathbf{h}$. Likewise, we can discretize $\mathbf{u}(\mathbf{x}) = \mathrm{DL}_\omega\mathbf{h}(\mathbf{x})$ to compute the solution at any $\mathbf{x}\in\mathbb{R}^2\backslash\overline{D}$. Precisely, we have

$$\mathbf{u}(\mathbf{x}) = \mathrm{DL}_\omega\mathbf{h}(\mathbf{x}) \approx \sum_{i=1}^{n/2}\sum_{k=1}^{3} {}^\omega\widetilde{A}_{(i,k),x}\mathbf{h}(\widetilde{\mathbf{v}}_{i,k}) =: \mathbf{u}_n(\mathbf{x})$$

with

$$\widetilde{A}_{(i,k),x} = \int_\sigma \left[\mathbf{T}_{\widetilde{\mathbf{m}}_i(s)}\left(\mathrm{K}_\omega(x,\widetilde{\mathbf{m}}_i(s))\right)\right]^\top \widetilde{J}_i(s)\widetilde{L}_k(s)\,\mathrm{d}s(s) \in \mathbb{C}^{2\times 2}\,.$$

Example 1. Consider the solution of the Navier equation $\Delta^*\mathbf{u}+\omega^2\mathbf{u}=0$ in $\mathbb{R}^2\backslash\overline{D}$ with $\mathbf{u}=\mathbf{f}$ on $\partial\Omega$, where the boundary of the domain Ω is given parametrically by $\mathbf{p}(\theta) = (2\cos(\theta),\sin(\theta))$ (an ellipse). The Lamé parameters are chosen to be $\lambda = 1$ and $\mu = 1$. The frequency ω is given by 1 and i and we used $\alpha = (1-\sqrt{3/5})/2$. The first column of the fundamental solution with $\mathbf{y} = (0,0)^\top$ satisfies $\Delta^*\mathbf{u}+\omega^2\mathbf{u}=0$ and is used as a reference solution. The boundary function $\mathbf{f}$ is chosen to be the first column of the fundamental solution restricted to the given boundary. We compute the solution at $\mathbf{x} = (3,3)^\top$ using the double layer ansatz $\mathbf{u}(\mathbf{x}) = \mathrm{DL}_\omega\mathbf{h}(\mathbf{x})$ and test therefore the operator $\frac{1}{2}\mathrm{I}+\mathrm{D}_\omega$ since we need to compute

$$\frac{1}{2}\mathbf{h}+\mathrm{D}_\omega\mathbf{h} = \mathbf{f}$$

in order to obtain $\mathbf{h}$. In Table 1.1, we list the absolute error $e_n^{(\omega)} = \|\mathbf{u}-\mathbf{u}_n\|$ for various choices of n as well as the estimated order of convergence $\mathrm{EOC}^{(\omega)} = \log\left(e_n^{(\omega)}/e_{2n}^{(\omega)}\right)/\log(2)$. As we can see in Table 1.1, we obtain a convergence order

n	$e_n^{(1)}$	$\mathrm{EOC}^{(1)}$	$e_n^{(\mathrm{i})}$	$\mathrm{EOC}^{(\mathrm{i})}$
10	1.31039_{-4}		7.74688_{-6}	
20	2.93242_{-5}	2.16	2.06750_{-6}	1.91
40	2.00761_{-6}	3.87	5.85069_{-8}	5.14
80	2.60633_{-7}	2.95	7.61938_{-9}	2.94

Table 1.1 Numerical results to test the discretization of the double layer operator for $\omega = 1$ and $\omega = \mathrm{i}$.

of at least two.

In a similar fashion, we can solve the problem $\Delta^*\mathbf{u}+\omega^2\mathbf{u}=0$ in $\mathbb{R}^2\backslash\overline{D}$ with the boundary conditions $\mathbf{u}=\mathbf{f}$, where $\mathbf{f}$ is a given function defined on the boundary.

The frequency ω is given as well. Using the single layer ansatz $\mathbf{u} = \mathrm{SL}_\omega \mathbf{g}$ in $\mathbb{R}^2 \backslash \overline{D}$ yields the boundary integral equation of the first kind

$$\mathrm{S}_\omega \mathbf{g} = \mathbf{f}. \tag{1.11}$$

Using the same strategy as explained before yields the linear system of size $3n \times 3n$

$$\sum_{i=1}^{n/2} \sum_{k=1}^{3} {}^{\omega}B_{(i,k),(j,\ell)} \mathbf{g}(\widetilde{\mathbf{v}}_{i,k}) = \mathbf{f}(\widetilde{\mathbf{v}}_{j,\ell}) \tag{1.12}$$

with

$${}^{\omega}B_{(i,k),(j,\ell)} = \int_\sigma \mathrm{K}_\omega(\widetilde{\mathbf{v}}_{j,\ell}, \widetilde{\mathbf{m}}_i(s)) \widetilde{J}_i(s) \widetilde{L}_k(s)\, \mathrm{d}s(s) \in \mathbb{C}^{2\times 2}$$

since the $(i,k),(j,\ell)$-entry is a 2×2 matrix. All four elements of the 2×2 matrix are

$$
\begin{aligned}
{}^{\omega}B^{(1,1)}_{(i,k),(j,\ell)} &= \int_\sigma u^{(1,1)}_{i,j,\ell}(s) \widetilde{J}_i(s) \widetilde{L}_k(s)\, \mathrm{d}s(s)\,, & {}^{\omega}B^{(1,2)}_{(i,k),(j,\ell)} &= \int_\sigma u^{(1,2)}_{i,j,\ell}(s) \widetilde{J}_i(s) \widetilde{L}_k(s)\, \mathrm{d}s(s) \\
{}^{\omega}B^{(2,1)}_{(i,k),(j,\ell)} &= \int_\sigma u^{(2,1)}_{i,j,\ell}(s) \widetilde{J}_i(s) \widetilde{L}_k(s)\, \mathrm{d}s(s)\,, & {}^{\omega}B^{(2,2)}_{(i,k),(j,\ell)} &= \int_\sigma u^{(2,2)}_{i,j,\ell}(s) \widetilde{J}_i(s) \widetilde{L}_k(s)\, \mathrm{d}s(s)
\end{aligned}
$$

with

$$
\begin{aligned}
u^{(1,1)}_{i,j,\ell}(s) &= \frac{\mathrm{i}}{4\mu} H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \\
&+ \frac{\mathrm{i}}{4\omega^2} \frac{k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) - k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|)}{\|\mathbf{d}_{i,j,\ell}(s)\|} \\
&+ \frac{(d^{(1)}_{i,j,\ell}(s))^2}{\|\mathbf{d}_{i,j,\ell}(s)\|^2} \left(\frac{\mathrm{i}}{4\omega^2} \left[k_p^2 H_0^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) - k_s^2 H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right. \\
&+ \left. \frac{\mathrm{i}}{2\omega^2 \|\mathbf{d}_{i,j,\ell}(s)\|} \left[k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) - k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
u^{(2,1)}_{i,j,\ell}(s) &= \frac{d^{(1)}_{i,j,\ell}(s) d^{(2)}_{i,j,\ell}(s)}{\|\mathbf{d}_{i,j,\ell}(s)\|^2} \left(\frac{\mathrm{i}}{4\omega^2} \left[k_p^2 H_0^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right. \right. \\
&- \left. k_s^2 H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right] + \frac{\mathrm{i}}{2\omega^2 \|\mathbf{d}_{i,j,\ell}(s)\|} \left[k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right. \\
&- \left. \left. k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right) \\
u^{(1,2)}_{i,j,\ell}(s) &= \frac{d^{(1)}_{i,j,\ell}(s) d^{(2)}_{i,j,\ell}(s)}{\|\mathbf{d}_{i,j,\ell}(s)\|^2} \left(\frac{\mathrm{i}}{4\omega^2} \left[k_p^2 H_0^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right. \right. \\
&- \left. k_s^2 H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right] + \frac{\mathrm{i}}{2\omega^2 \|\mathbf{d}_{i,j,\ell}(s)\|} \left[k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right. \\
&- \left. \left. k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
u_{i,j,\ell}^{(2,2)}(s) = {} & \frac{\mathrm{i}}{4\mu} H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \\
& + \frac{\mathrm{i}}{4\omega^2} \frac{k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) - k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|)}{\|\mathbf{d}_{i,j,\ell}(s)\|} \\
& + \frac{(d_{i,j,\ell}^{(2)}(s))^2}{\|\mathbf{d}_{i,j,\ell}(s)\|^2} \left(\frac{\mathrm{i}}{4\omega^2} \left[k_p^2 H_0^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) - k_s^2 H_0^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right. \\
& \left. + \frac{\mathrm{i}}{2\omega^2 \|\mathbf{d}_{i,j,\ell}(s)\|} \left[k_s H_1^{(1)}(k_s \|\mathbf{d}_{i,j,\ell}(s)\|) - k_p H_1^{(1)}(k_p \|\mathbf{d}_{i,j,\ell}(s)\|) \right] \right)
\end{aligned}
$$

For a given function $\mathbf{f}$ and frequency ω, the linear system (1.12) is solved directly for $\mathbf{g}$. We discretize $\mathbf{u}(\mathbf{x}) = \mathrm{SL}_\omega \mathbf{g}(\mathbf{x})$ to compute the solution at any $\mathbf{x} \in \mathbb{R}^2 \backslash \overline{D}$. Precisely, we have

$$
\mathbf{u}(\mathbf{x}) = \mathrm{SL}_\omega \mathbf{g}(\mathbf{x}) \approx \sum_{i=1}^{n/2} \sum_{k=1}^{3} {}^{\omega}\widetilde{B}_{(i,k),x} \mathbf{g}(\widetilde{\mathbf{v}}_{i,k}) =: \mathbf{u}_n(\mathbf{x})
$$

with

$$
\widetilde{B}_{(i,k),x} = \int_\sigma \mathrm{K}_\omega(x, \widetilde{\mathbf{m}}_i(s)) \widetilde{J}_i(s) \widetilde{L}_k(s) \, \mathrm{d}s(s) \in \mathbb{C}^{2\times 2}.
$$

Example 2. Consider again the solution of the Navier equation $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0$ in $\mathbb{R}^2 \backslash \overline{D}$ with $\mathbf{u} = \mathbf{f}$ on $\partial\Omega$, where the boundary of the domain Ω is given parametrically by $\mathbf{p}(\theta) = (2\cos(\theta), \sin(\theta))$ (an ellipse). We use the same parameters as before (refer to Example 1). We again compute the solution at $\mathbf{x} = (3,3)^\top$, but with a single layer ansatz $\mathbf{u}(\mathbf{x}) = \mathrm{SL}_\omega \mathbf{g}(\mathbf{x})$ and test therefore the operator S_ω since we need to compute

$$
\mathrm{S}_\omega \mathbf{g} = \mathbf{f}
$$

to obtain $\mathbf{g}$. In Table 1.2, we list the absolute error $e_n^{(\omega)}$ for various choices of n including the estimated order of convergence $\mathrm{EOC}^{(\omega)}$. As we can see in Table 1.2,

n	$e_n^{(1)}$	$\mathrm{EOC}^{(1)}$	$e_n^{(\mathrm{i})}$	$\mathrm{EOC}^{(\mathrm{i})}$
10	1.47550_{-4}		1.31082_{-5}	
20	1.35472_{-5}	3.45	1.39689_{-6}	3.23
40	2.15646_{-7}	5.97	2.03561_{-8}	6.10
80	9.55428_{-8}	1.17	9.41391_{-9}	1.11

Table 1.2 Numerical results to test the discretization of the single layer operator for $\omega = 1$ and $\omega = \mathrm{i}$.

we obtain a convergence order of at least two.

1.4 Solving the non-linear eigenvalue problem

Beyn's algorithm [Be12] is used to solve the non-linear eigenvalue problem of the form

$$\mathrm{M}(\omega)\mathbf{u} = \mathbf{0}\,, \mathbf{u} \neq 0$$

with $\mathrm{M}(\omega) \in \mathbb{C}^{m\times m}$. Therefore, the user specifies a smooth contour γ in the complex plane and integrates over the resolvent. We will use a circle with radius R centered at c as the contour γ given parametrically by $\phi(t) = c + R\mathrm{e}^{t\,\mathrm{i}}$ and $\phi'(t) = R\mathrm{i}\mathrm{e}^{t\,\mathrm{i}}$. With Keldysh's theorem one can reduce the non-linear eigenvalue problem to a linear eigenvalue problem of size $n(\gamma)$ which is much smaller than m. To be more specific, one has to compute the two integrals

$$\mathrm{A}_0 = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma \mathrm{M}^{-1}(\omega)\hat{\mathrm{V}}\,\mathrm{d}s(\omega)\,, \qquad \mathrm{A}_1 = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma \omega\mathrm{M}^{-1}(\omega)\hat{\mathrm{V}}\,\mathrm{d}s(\omega)\,,$$

where $\hat{\mathrm{V}} \in \mathbb{C}^{m\times\ell}$ with $m \gg \ell \geq n(\gamma)$ is a random matrix. The parameter ℓ has to be chosen such that is is greater than the number of possible eigenvalues $n(\gamma)$ (including multiplicities), but as small as possible to reduce computational work. Of course, the two integrals have to be computed numerically. We will use the trapezoidal rule yielding

$$\mathrm{A}_{0,N} = \frac{1}{\mathrm{i}N}\sum_{j=0}^{N-1}\mathrm{M}^{-1}(\phi(t_j))\hat{\mathrm{V}}\phi'(t_j)\,, \quad \mathrm{A}_{1,N} = \frac{1}{\mathrm{i}N}\sum_{j=0}^{N-1}\phi(t_j)\mathrm{M}^{-1}(\phi(t_j))\hat{\mathrm{V}}\phi'(t_j)\,.$$

The parameter N is specified by the user and with this we define the equidistant nodes $t_j = 2\pi j/N,\ j = 0,\ldots,N$. The parameter N can be chosen small since the trapezoidal rule converges exponentially. Next, a (reduced) singular value decomposition of $\mathrm{A}_{0,N} = \mathrm{V}\Sigma\mathrm{W}^{\mathrm{H}}$ is computed, where $\mathrm{V} \in \mathbb{C}^{m\times\ell}$, $\Sigma \in \mathbb{C}^{\ell\times\ell}$, and $\mathrm{W} \in \mathbb{C}^{\ell\times\ell}$. Then, a rank test on the diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_\ell)$ is performed which indicates how many eigenvalues including multiplicities are contained within the chosen contour γ. We will use $\varepsilon = 10^{-2}$ and compute $n(\gamma)$ such that $\sigma_1 \geq \ldots \geq \sigma_{n(\gamma)} > \varepsilon > \sigma_{n(\gamma)+1} \geq \ldots \geq \sigma_\ell$ is satisfied. With this, we construct the three matrices $\mathrm{V}_0 = (\mathrm{V}_{ij})_{1\leq i\leq m, 1\leq j\leq n(\gamma)}$, $\Sigma_0 = (\Sigma_{ij})_{1\leq i\leq n(\gamma), 1\leq j\leq n(\gamma)}$, and $\mathrm{W}_0 = (\mathrm{W}_{ij})_{1\leq i\leq \ell, 1\leq j\leq n(\gamma))}$. Finally, we compute $n(\gamma)$ eigenvalues, say ω_i, and eigenvectors $\mathbf{s}_i$ of the new matrix $\mathrm{B} = \mathrm{V}_0^{\mathrm{H}}\mathrm{A}_{1,N}\mathrm{W}_0\Sigma_0^{-1} \in \mathbb{C}^{n(\gamma)\times n(\gamma)}$. The i-th non-linear eigenvector $\mathbf{u}_i$ is given by $\mathrm{V}_0\mathbf{s}_i$.

1.5 Numerical results

In this section, we present numerical results for the computation of elastic interior transmission eigenvalues for a variety of two-dimensional domains. Let $\theta \in [0, 2\pi]$. The first domain D_1 under consideration is a disk with radius $r_1 = 1/2$ having the parametrization $\mathbf{p}_1(\theta) := (r_1\cos(\theta), r_1\sin(\theta))^\top$. The second domain D_2 is an ellipse with semi-axis $a_2 = 1$ and $b_2 = 1/2$. Its parametrization is given by $\mathbf{p}_2(\theta) := (a_2\cos(\theta), b_2\sin(\theta))^\top$. The third parametrization is given by $\mathbf{p}_3(\theta) := (3\cos(\theta)/4 + 3\cos(2t)/10, \sin(\theta))^\top$ and represents the 'deformed ellipse' (kite) domain D_3. The unit square D_4 is the fourth domain under consideration.

For comparison, we will use the parameters $\rho_1 = 1$, $\rho_2 = 4$ and the Lamé parameters $\mu = 1/16$ and $\lambda = 1/4$ which have been used in a variety of papers before. Further, we use $N = 24$, $\ell = 20$, $\varepsilon = 10^{-2}$, and $R = 1/4$ within the Beyn algorithm. The parameter c and the number of faces n_f depend on the considered domain and are listed separately. The parameter α is chosen to be $(1-\sqrt{3/5})/2$ for all the following numerical results.

At first, we consider D_1 and compute the first seven real elastic interior transmission eigenvalues using $n_f = 40$ and $c = 1.5$ for ω_1, ω_2, and ω_3 and $n_f = 40$ and $c = 2.1$ for ω_4, ω_5, ω_6, and ω_7 with the boundary element method (BEM). We compare our results with the method of fundamental solutions (MFS) [KlPi20] since those results are accurate up to ten digits accuracy for D_1. Additionally, we compare our results with different finite element methods (FEM) [YaEtAl20, XiJi18, JiLiSu18]. Note that the second, fourth, and sixth eigenvalue have multiplicity two. In [XiJi18] the first two eigenvalues are listed and in [YaEtAl20, JiLiSu18] the first six eigenvalues are computed. In Table 1.3, we list the first seven eigenvalues and highlight the correct number of digits in bold. The eigenvalues obtained with the MFS are used for comparison. All reported digits are correct and therefore not highlighted in bold.

ITE	BEM	FEM [YaEtAl20]	FEM [XiJi18]	FEM [JiLiSu18]	MFS [KlPi20]
ω_1	**1.451 304**	**1.45**2 482	**1.451** 948	**1.45**5 078	1.451 304 028
ω_2	**1.704 6**45	**1.70**6 023	**1.70**5 370	**1.70**9 214	1.704 638 247
ω_3	**1.704 6**45	**1.70**6 023		**1.70**9 214	
ω_4	**1.984 5**51	**1.98**6 143		**1.98**9 630	1.984 530 256
ω_5	**1.984 5**52	**1.98**6 146		**1.98**9 630	
ω_6	**2.269 1**52	**2.2**70 963		**2.2**74 992	2.269 112 085
ω_7	**2.269 1**52				

Table 1.3 Numerical results for the first seven real elastic interior transmission eigenvalues for a disk with radius 1/2.

As we can see, our numerical results are accurate up to five digits accuracy using only $n_f = 40$ faces. The first eigenvalue is accurate up to six digits. The numerical results for the FEM methods are only accurate up to two to three digits with the exception of the first eigenvalue which is accurate up to four digits. The used mesh size in [XiJi18] is $h = 1/160$, in [JiLiSu18] is $h = 1/80$, and in [YaEtAl20] is $h \approx 0.03125$. Note that in the preprint [XiJiGe18] $h = 0.0125$ was used and yields **1.45**6 for the first eigenvalue. In sum, our numerical results are much more accurate than the ones given by FEM. However, the best results are given by the MFS.

Next, we consider the ellipse D_2. We use $n_f = 40$ and $c = 1.4$ and compare our numerical results given in Table 1.4 with the MFS for the first four real interior transmission eigenvalues. The numerical results of the MFS are accurate with ten digits and serve again as reference values. They are not highlighted in bold. Unfortunately, no numerical results are available for the FEM method.

ITE	BEM	MFS [KlPi20]
ω_1	**1.296** 681	1.296 728 137
ω_2	**1.302** 814	1.302 785 814
ω_3	**1.540** 775	1.540 896 035
ω_4	**1.5651** 73	1.565 151 107

Table 1.4 Numerical results for the first four real elastic interior transmission eigenvalues for an ellipse with semi-axis 1 and $1/2$.

As we can see, we are able to obtain four digits accuracy. The fourth eigenvalue is accurate up to five digits accuracy. All eigenvalues are simple. Hence, the BEM method is a good alternative for the MFS and offers good flexibility in terms of using general domains. This is shown with the next domain D_3.

The numerical results for the first four elastic interior transmission eigenvalues for the kite are given in Table 1.5 using $n_f = 40$ faces and $c = 0.9$ for ω_1 and $n_f = 40$ and $c = 1.1$ for ω_2, ω_3, and ω_4 along with the numerical results obtained with the MFS. The eigenvalues obtained with the MFS are correct to four digits accuracy and not highlighted in bold.

ITE	BEM	MFS [KlPi20]
ω_1	**0.947** 495	0.947
ω_2	**1.047 3**98	1.047
ω_3	**1.111** 190	1.111
ω_4	**1.235** 261	1.235

Table 1.5 Numerical results for the first four real elastic interior transmission eigenvalues for the kite domain.

We obtain at least four digits accuracy with the BEM for ω_1, ω_2, ω_3, and ω_4. For ω_2 we obtain five digits accuracy. Hence, the results are equal or better than the ones of the MFS. Therefore, the BEM method offers the flexibility to use it for more general domains with a smooth boundary. Unfortunately, no numerical results are reported with the FEM for such domains.

Of course, the FEM is much better suited for polygonal domains such as the unit square. We finally compare our method with the FEM (the accuracy is not known, but at least five digits) and the MFS (five digits accuracy). We use $n_f = 46$ and $c = 1.5$ for the first eigenvalue and $n_f = 46$ and $c = 1.8$ for the other eigenvalues to obtain the numerical results that are given in Table 1.6.

Our numerical results are better than the ones given in [XiJi18] ($h = 0.00625$) and [XiJiGe18] ($h = 0.0125$). Moreover, are the results comparable with the MFS. However, the numerical results reported in [YaEtAl20] ($h \approx 0.03125$) and [JiLiSu18]

ITE	BEM	FEM [YaEtAl20]	FEM [JiLiSu18]	FEM [XiJiGe18]	FEM [XiJi18]	MFS [KlPi20]
ω_1	**1.393 8**92	1.393 877	1.393 874	1.393 879	**1.39**4 419	**1.393 8**
ω_2	**1.618 2**64	1.618 299	1.618 296		**1.61**9 008	**1.618 2**
ω_3	**1.618** 389	1.618 299	1.618 296			
ω_4	**1.802 0**89	1.802 042	1.802 032			**1.802 0**
ω_5	**1.936 1**87	1.936 138	1.936 134			**1.936** 2

Table 1.6 Numerical results for the first five real elastic interior transmission eigenvalues for the unit square.

($h \approx 0.025$) are better as expected. The same is true for FEM [YaHaBi20] using $m = 26$.

Finally, note that we can easily compute complex-valued elastic interior transmission eigenvalues by selecting a corresponding contour in the complex plane, although the existence of them is still an open question. Using $c = 2 + \mathrm{i}/2$ and $n_f = 40$ for D_1 and $n_f = 46$ for D_4 yields the results reported in Table 1.7.

Domain	BEM
Circle	$1.987\,189 + 0.283\,146\mathrm{i}$
Unit square	$1.865\,629 + 0.291\,766\mathrm{i}$

Table 1.7 Numerical results for one complex-valued elastic interior transmission eigenvalues for the circle with radius $1/2$ and the unit square.

1.6 Summary and outlook

We presented an algorithm to compute interior elastic transmission eigenvalues in two dimensions with the boundary element collocation method in combination with a non-linear eigenvalue solver. We are able to obtain good results for a circle and an ellipse which outperforms various finite elements methods. However, the method of fundamental solutions beats the boundary element method in accuracy. The situation is different for polygonal domains such as a square. The best method in accuracy is the finite element method. However, for various domains with a smooth boundary, the boundary element method is the one for which the best accuracy can be obtained.

The python program is available at

`https://github.com/kleefeld80/elastic-ite-bem`

and has been developed and tested under Windows 10 with python version 3.8. All numerical results reported within this chapter have been obtained with python version 3.9.4 under Windows 10 and can be reproduced using the `runall.py` script.

No numerical results are reported for the three-dimensional case. Hence, the next step would be to use the presented algorithm to numerically calculate interior elastic transmission eigenvalues in three dimensions with the boundary element method in a similar fashion as presented in [Kl13] for interior acoustic transmission eigenvalues.

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