Extracting the low-energy constant L_0^r at three flavors from pion-kaon scattering

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Abstract

Based on our analysis of the contributions from the connected and disconnected contraction diagrams to the pion-kaon scattering amplitude, we provide the first determination of the only free low-energy constant at $\mathcal{O}(p^4)$, known as L_0^r, in SU(4|1) Partially-Quenched Chiral Perturbation Theory using the data from the Extended Twisted Mass Collaboration, $L_0^r(\mu = M_\rho) = 0.77(20)(25)(7)(7)(2) \cdot 10^{-3}$. The theory uncertainties originate from the unphysical scattering length, the physical low-energy constants, the higher-oder chiral corrections, the (lattice) meson masses and the pion decay constant, respectively.

1. Introduction

Partially-Quenched Quantum Chromodynamics (PQQCD) and its low-energy effective field theory (EFT), Partially-Quenched Chiral Perturbation Theory(PQChPT) [1, 2, 3, 4, 5, 6, 7], are powerful tools to assist the first-principles calculations of hadronic observables using lattice QCD. They were originally created to handle the so-called partially-quenched approximation in early lattice studies, where the "valence" and "sea" quark masses were made distinct in order to simplify the calculation of the fermion determinant. Nowadays such an approximation has been largely abandoned, but the devised theory frameworks have found their own ways to continue being useful. In particular, it was recently realized that since additional quark flavors are introduced in PQChPT, it allows an EFT description of each individual quark contraction diagram in the lattice simulation of a given physical observable, which is very useful for getting a better handle on the noisier and computationally-expensive contractions (the so-called "disconnected diagrams"). This idea was applied initially to the study of the hadronic vacuum polarization [8] and the pion scalar form factor [9], and was later extended to pion-pion scattering [10, 11] and the parity-odd pion-nucleon coupling constant [12].

As in any EFT, the full predictive power of PQChPT to a given order is guaranteed only when all the low-energy constants (LECs) at that order are fixed. This is a non-trivial task since some of them are not constrained by any physical experiment and can only be determined through lattice simulations. In particular, in an extended flavor sector with $N_f \geq 4$, the Cayley-Hamilton relation is no longer valid, which leads to the following term in the PQChPT Lagrangian at $\mathcal{O}(p^4)$:

$$\mathcal{L}^{(4)} = L_0 \,\mathrm{Str}\left(\partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U\right) \,, \tag{1}$$

where 'Str' denotes the supertrace over the extended flavor space, U is the standard exponential representation of the pseudo-Nambu-Goldstone bosons, and L_0 is a new LEC that does not appear in ordinary two-flavor and three-flavor ChPT. Despite appearing at next-to-leading order (NLO), this LEC contributes to static quantities such as the pion mass and decay constant only at the next-to-next-to-leading order (NNLO), when the valence and sea quark masses are kept distinct. Based on this, Ref. [13] has determined the renormalized LEC L_0^r in the SU(4|2) PQChPT, which is equivalent to a two-flavor ChPT

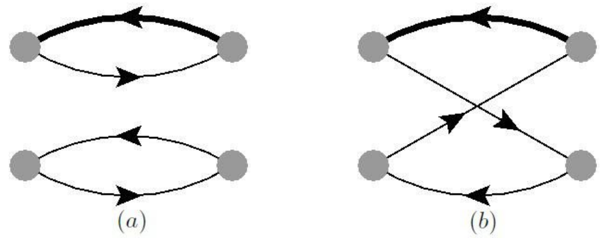

Figure 1: Connected contraction diagrams T_a and T_b. The thick line indicates an $\langle s\bar{s}\rangle$ contraction.

in computations of physical processes. The quoted result is $L_0^r(\mu = 1\ \text{GeV}) = 1.0(1.1)\cdot 10^{-3}$, using dimensional regularization.

Refs. [10, 11] provided an alternative determination of L_0^r in SU(4|2) using the fact that it appears in separate contraction diagrams in the $\pi\pi$ scattering amplitude at NLO, and therefore can be obtained from the unphysical scattering length defined through an appropriate linear combination of contraction diagrams. The advantage of this new method is that, since only light quarks are involved in the procedure, the higher-order chiral corrections that scale generically as $M_\pi^2/(4\pi F_\pi)^2$ are expected to be small so the NLO fitting is more stable than NNLO, and of course the number of unknown LECs in the former is also smaller. Using the lattice data of connected $\pi\pi$ correlation functions by the Extended Twisted Mass (ETM) Collaboration [14], Ref. [11] reported $L_0^r(\mu = 1\ \text{GeV}) = 5.7(1.9)\cdot 10^{-3}$. The significant difference with the result from the NNLO fit in Ref. [13] is yet to be understood. Resolving such a discrepancy may improve our knowledge of the chiral dynamics at low energies, or even reveal some unexpected lattice systematics that could play important roles in precision physics.

The analysis in Refs. [10, 11] could be straightforwardly generalized to the three-flavor case. In particular, the so-called SU(4|1) PQChPT is of special interest because it is the simplest extension of the original three-flavor ChPT. Moreover, among all its renormalized LECs at $\mathcal{O}(p^4)$, only L_0^r is undetermined, while all the others are identical to those in SU(3). This implies that the theory would be fully predictive at $\mathcal{O}(p^4)$ once L_0^r is fixed, and could then be used to aid the lattice studies of interesting hadronic processes involving strange quarks, such as $K\pi\to K\pi$, $\pi\pi\to K\bar{K}$ and $K\eta\to K\eta$ scatterings, in particular channels where disconnected diagrams appear (e.g. the $I = 1/2$ channel of $K\pi$ scattering).

In this letter, we present the first-ever numerical determination of L_0^r in SU(4|1) based on the method outline in our previous paper, Ref. [15]. The main idea is that, by switching the relative sign between the two types of connected contraction diagrams that occur in the $I = 3/2$ $K\pi$ scattering, one obtains effectively an unphysical single-channel scattering amplitude T_β, of which the scattering length depends on L_0^r at $\mathcal{O}(p^4)$. Invoking the recent lattice data by the ETM collaboration [16], the unphysical scattering length is obtained through the usual Lüscher analysis of the discrete energy states [17], which consequently fixes L_0^r. This completes the SU(4|1) PQChPT Lagrangian at $\mathcal{O}(p^4)$ and is potentially useful for all the purposes mentioned above. Besides, it serves as a prototype for future analyses of more complicated PQ-extensions of three-flavor ChPT. Such theories, after integrating out the strange quark, reduce to PQChPT with only light flavors. This may give an independent check of value of L_0^r in SU(4|2), and provide some hints towards the solution to the aforementioned discrepancy.

2. Formalism

As detailed in Ref. [15], the pertinent two contractions in $K\pi$ scattering with total isospin $I = 3/2$, as depicted in Fig.1, are related to the amplitudes in the SU(4|1) PQChPT:

$$\begin{aligned} T_a(s,t,u) &= T_{(u\bar{s})(d\bar{j})\to(u\bar{s})(d\bar{j})}(s,t,u)\ , \\ T_b(s,t,u) &= T_{(u\bar{s})(d\bar{j})\to(d\bar{s})(u\bar{j})}(s,t,u)\ , \end{aligned} \tag{2}$$

with s,t,u the conventional Mandelstam variables subject to the constraint $s+t+u = 2(M_K^2+M_\pi^2)$, u,d,s are the physical quarks and j denotes the additional valence quark (which comes together with a ghost quark $\tilde{j}$) in PQChPT. The assigned quark masses are $m_u = m_d = m_j = m_{\tilde{j}} < m_s$. For total isospin $I = 1/2$, there is one additional scattering amplitude T_c that is related to T_b through a simple crossing, $T_c(s,t,u) \equiv T_b(u,t,s)$. Although such a crossing is analytically straightforward, from a lattice point of view T_a and T_b are relatively easy to evaluate, whereas T_c involves a pair of quark propagators that start and end on the same temporal slice, and is exactly what we call a "disconnected diagram". Such diagram suffers from a low signal-to-noise ratio, and represents a fundamental challenge in the first-principles study of the $K\pi$ scattering in the $I = 1/2$ channel.

From the above one may define three effective single-channel scattering amplitudes:

$$\begin{aligned} T_\alpha(s,t,u) &= T_a(s,t,u) + T_b(s,t,u) \\ T_\beta(s,t,u) &= T_a(s,t,u) - T_b(s,t,u) \\ T_\gamma(s,t,u) &= T_a(s,t,u) - \frac{1}{2}T_b(s,t,u) + \frac{3}{2}T_c(s,t,u) \ , \end{aligned} \tag{3}$$

where T_α and T_γ correspond to the physical $I = 3/2$ and $I = 1/2$ scattering amplitudes, respectively, while T_β is an unphysical amplitude. The S-wave scattering lengths are defined through the threshold values of the amplitudes:

$$a_0^i = -\frac{1}{8\pi\sqrt{s_0}} T_i(s_0, t_0, u_0) \quad (i = \alpha, \beta, \gamma) \ , \tag{4}$$

with $s_0 = (M_K + M_\pi)^2$, $t_0 = 0$ and $u_0 = (M_K - M_\pi)^2$. Obviously, only the unphysical scattering length a_0^β can depend on the unphysical LEC L_0^r. Its explicit expression at $\mathcal{O}(p^4)$ reads:

$$\begin{aligned} a_0^\beta &= \frac{-1}{8\pi\sqrt{s_0}}\Bigg[\frac{M_\pi^2 M_K^2}{F_\pi^4}\left(-96L_0^r + 32L_1^r + 32L_2^r - 16L_3^r - 32L_4^r + 8L_5^r + 32L_6^r - 16L_8^r - \frac{5}{576\pi^2}\right) \\ &+ \frac{8M_\pi^3 M_K L_5^r}{F_\pi^4} + \frac{\mu_\pi}{F_\pi^2\,(M_K^2 - M_\pi^2)}\left(-\frac{M_K^4}{4} - \frac{1}{2}M_\pi M_K^3 + \frac{13}{4}M_\pi^2 M_K^2 + \frac{5}{2}M_\pi^3 M_K\right) \\ &+ \frac{\mu_K}{F_\pi^2}\left(-\frac{M_K^3}{M_K - M_\pi} - \frac{M_\pi M_K^2}{M_K - M_\pi} - \frac{4M_\pi^2 M_K^2}{M_K^2 - M_\pi^2} - \frac{M_\pi^2 M_K}{2\,(M_K - M_\pi)} + \frac{M_\pi^3}{2\,(M_K - M_\pi)}\right) \\ &+ \frac{\mu_\eta}{F_\pi^2\,(M_K^2 - M_\pi^2)}\left(-\frac{4M_\pi^2 M_K^4}{9M_\eta^2} - \frac{M_\pi^4 M_K^2}{18M_\eta^2} + \frac{3M_K^4}{4} + \frac{3}{2}M_\pi M_K^3 + \frac{11}{4}M_\pi^2 M_K^2 - \frac{3}{2}M_\pi^3 M_K\right) \\ &+ \frac{M_\pi^2 M_K^2 \bar{J}_{\pi K}(s_0)}{F_\pi^4} + \frac{\bar{J}_{\pi K}(u_0)}{F_\pi^4}\left(\frac{M_K^4}{8} + \frac{1}{2}M_\pi M_K^3 - \frac{1}{4}M_\pi^2 M_K^2 + \frac{1}{2}M_\pi^3 M_K + \frac{M_\pi^4}{8}\right) \\ &+ \frac{\bar{J}_{K\eta}(u_0)}{F_\pi^4}\left(\frac{M_K^4}{72} + \frac{1}{18}M_\pi M_K^3 - \frac{7}{12}M_\pi^2 M_K^2 + \frac{1}{18}M_\pi^3 M_K + \frac{M_\pi^4}{72}\right) \\ &+ \frac{\bar{\bar{J}}_{\pi K}\,(u_0)}{F_\pi^4}\left(-\frac{M_K^4}{8} - \frac{1}{2}M_\pi M_K^3 - \frac{3}{4}M_\pi^2 M_K^2 - \frac{1}{2}M_\pi^3 M_K - \frac{M_\pi^4}{8}\right) \\ &+ \frac{\bar{\bar{J}}_{K\eta}\,(u_0)}{F_\pi^4}\left(-\frac{M_K^4}{72} - \frac{1}{18}M_\pi M_K^3 - \frac{1}{12}M_\pi^2 M_K^2 - \frac{1}{18}M_\pi^3 M_K - \frac{M_\pi^4}{72}\right) - \frac{5M_K^4}{384\pi^2 F_\pi^4} - \frac{5M_\pi M_K^3}{192\pi^2 F_\pi^4} \\ &- \frac{M_\pi^3 M_K}{192\pi^2 F_\pi^4} + \frac{M_\pi M_K}{F_\pi^2} - \frac{M_\pi^4}{384\pi^2 F_\pi^4}\Bigg] \ , \end{aligned} \tag{5}$$

where $\mu_P = (M_P^2/32\pi^2 F_\pi^2)\ln(M_P^2/\mu^2)$, with μ the renormalization scale, and the functions $\bar{J}_{PQ}(s)$, $\bar{\bar{J}}_{PQ}(s)$ are given in Ref. [15]. The $\{L_i^r\}$ are the $\mathcal{O}(p^4)$ LECs in SU(4|1) PQChPT, among which L_{1-8}^r are numerically identical with those in the ordinary three-flavor ChPT. Only L_0^r is new, and can be readily solved from the equation above.

3. Extraction of L_0^r

The unphysical scattering length a_0^β is an essential input in our study. It is obtained from the analysis of the lattice-volume-dependence of the discrete energy levels corresponding to the combination $T_a - T_b$

Ensemble	aM_π	aM_K	aM_η	aF_π
A30.32	0.1239(3)	0.236(7)	0.314(17)	0.06452(21)
A40.24	0.1452(5)	0.241(7)	0.317(7)	0.06577(24)

Table 1: Basic parameters of the two lattice ensembles in Ref.[16], with lattice spacing $a = 0.0885(36)$ fm. The uncertainties are mainly from finite-volume effects.

$\mu_{\pi K} a_0^\beta$		
Ensemble	**E1**	**E2**
A30.32	$-0.0956(91)$	$-0.0961(84)$
A40.24	$-0.1152(144)$	$-0.1142(131)$

Table 2: The unphysical scattering length a_0^β evaluated from the ensembles A30.32 and A40.24. The uncertainties are mainly from thermal pollutions, which are treated using two different methods (**E1** and **E2**).

through the standard Lüscher formula [17]. In this work, we utilize the results in Ref. [16] for the $K\pi$ system in the $I = 3/2$ channel where the correlation functions $C_a(\tau)$ and $C_b(\tau)$ that correspond to the two contractions in Fig.1 were separately calculated. The calculation was based on the $N_f = 2+1+1$ twisted mass lattice QCD. For our analysis we have considered the ensembles A30.32 and A40.24 for the determination of the discrete ground-state energies of the $K\pi$ system. The basic parameters of each ensemble are summarized in Table 1.

Only results that correspond to the physical combination $C_a(\tau)+C_b(\tau)$ were displayed in Ref. [16] for obvious reasons. In this project, we acquired the unphysical S-wave scattering length a_0^β directly from the authors of that paper, who obtained its value through an unpublished analysis of the volume-dependence of the discrete energy levels extracted from the unphysical combination $C_\beta(\tau) = C_a(\tau) - C_b(\tau)$ [18]: First, the energy shift $\delta E_\beta = E_\beta^{\pi K} - M_\pi - M_K$ was obtained as a function of the lattice size L from the exponential behavior of $C_\beta(\tau)$ at large Euclidean time τ. The scattering length a_0^β at infinite volume was then computed using the single-channel, Taylor-expanded Lüscher formula

$$\delta E_\beta = -\frac{2\pi a_0^\beta}{\mu_{\pi K} L^3}\left(1 + c_1 \frac{a_0^\beta}{L} + c_2 \frac{(a_0^\beta)^2}{L^2}\right) + \mathcal{O}(L^{-6}) \tag{6}$$

as in Eq.(14) of Ref. [16]. Here, $\mu_{\pi K}$ is the reduced mass of the $K\pi$ system and $c_{1,2}$ are known coefficients. The final outcomes are summarized in Table 2. The main uncertainty of a_0^β is systematic, which comes from the unwanted time-dependent contributions at finite τ (i.e. the "thermal pollutions"). Their effects were studied using two different methods labeled as **E1** (weighting and shifting) and **E2** (dividing out the pollution), respectively [19].

To solve for L_0^r using Eq. (5), we further require the values of all the physical LECs. We took their values at $\mu = M_\rho = 770$ MeV from Ref. [20]:

$$\begin{aligned} 10^3 L_1^r &= 1.11(10)\ ,\quad 10^3 L_2^r = 1.05(17)\ ,\quad 10^3 L_3^r = -3.82(30)\ ,\quad 10^3 L_4^r = 1.87(53)\ , \\ 10^3 L_5^r &= 1.22(06)\ ,\quad 10^3 L_6^r = 1.46(46)\ ,\quad 10^3 L_8^r = 0.65(07)\ . \end{aligned} \tag{7}$$

With all the above, we may now compute L_0^r straightforwardly. The outcome at $\mu = M_\rho$ reads:

$$\begin{aligned} &\text{A30.32}\ ,\ \text{Method E1} \quad : \quad 10^3 L_0^r = 0.78(21)(25)(7)(7)(2) \\ &\text{A30.32}\ ,\ \text{Method E2} \quad : \quad 10^3 L_0^r = 0.77(20)(25)(7)(7)(2) \end{aligned}$$

$$\begin{aligned} &\text{A40.24 , Method E1} \quad : \quad 10^3 L_0^r = 0.86(25)(25)(7)(4)(2) \\ &\text{A40.24 , Method E2} \quad : \quad 10^3 L_0^r = 0.87(22)(25)(7)(4)(2) \end{aligned} \tag{8}$$

where the uncertainties come from a_0^β, the physical LECs, the higher-order ChPT corrections, the (lattice) meson masses and F_π, respectively. In particular, the higher-order ChPT corrections are estimated by multiplying the central value with the usual chiral suppression factor $M_K^2/(4\pi F_\pi)^2$. We see that the values of L_0^r from all four determinations are consistent with each other within the error bars, so we may simply quote the number with the smallest theory uncertainty, namely the one from the ensemble A30.32 with method **E2**:

$$L_0^r(M_\rho) = 0.77(33) \cdot 10^{-3} \ . \tag{9}$$

Finally, we comment on the relation between this result and the corresponding LEC in SU(4|2). In principle, relations between LECs in the two-flavor and three-flavor ChPT can be obtained by integrating out the strange quark in the latter. Possible PQChPT extensions of an ordinary two-flavor ChPT are SU(3|1), SU(4|2), SU(5|3)..., etc, but only SU(4|2) onwards possess an L_0-dependence at tree-level as it requires at least four fermionic quarks. Similarly, possible PQChPT extensions of a three-flavor ChPT are SU(4|1), SU(5|2), SU(6|3)... . In Ref. [15] we chose the simplest version which is SU(4|1). After integrating the strange quark, it reduces to SU(3|1) that does not depend on L_0 at tree-level. Therefore, it is not possible to discuss the matching between L_0 in two- and three-flavors based on the theory setup in Ref. [15]. For that, one would have to repeat the calculations using a larger graded algebra, such as SU(5|2). This is of great interest because it may provide new insights to the apparent disagreement between the determination of L_0^r at SU(4|2) from NLO and NNLO. However, it goes beyond the scope of this letter and will be carried out in follow-up studies.

4. Summary

In this work, we have for the first time determined the unphysical LEC $L_0^r(M_\rho)$ in the simplest PQ-extension of the three-flavor ChPT through an NLO analysis of contraction diagrams in $K\pi$ scattering, $L_0^r = 0.77(33) \cdot 10^{-3}$. Utilizing the precise data from the ETM collaboration for $K\pi$ scattering in the $I = 3/2$ channel, we control the absolute uncertainty in this LEC to 3.3×10^{-4}, which is better than the previous determinations of the corresponding LEC for two flavors in Ref. [13] that made use of an NNLO fitting, and in the NLO fitting to the $\pi\pi$ scattering amplitudes in Refs. [10, 11] that depends on more unknown LECs. The major sources of uncertainty in this study are the systematic errors in the lattice extraction of the unphysical scattering length a_0^β, as well as the physical LECs L_{1-8}^r. Our work completes the PQChPT Lagrangian at $\mathcal{O}(p^4)$ and prepares it for the future applications in studies of interesting hadronic observables involving strange quarks, in synergy with lattice QCD.

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