Definition of Local Spatial Densities in Hadrons

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We show that the matrix element of a local operator between hadronic states can be used to unambiguously define the associated spatial density. As an explicit example, we consider the charge density of a spinless particle and clarify its relationship to the electric form factor. Our results lead to an unconventional interpretation of the spatial densities of local operators and their moments.

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Introduction.—It is often claimed that the electric charge density of the nucleon is given by the three-dimensional Fourier transform of its electric form factor in the Breit frame [1–3]. Similar relations have been suggested for Fourier transforms of gravitational form factors and various local distributions in Refs. [4–6].

The identification of spatial density distributions with the Fourier transform of the corresponding form factors for systems whose intrinsic size is comparable with the Compton wavelength was criticized in Refs. [7–14]. In particular, Miller pointed out that the derivation of the conventional relationship between the charge density and the electric form factor in the Breit frame by Sachs [3] implicitly assumes *delocalized* wave packet states [12]. This would result in moments of the charge distribution governed by the size of the wave packet rather than the intrinsic properties of the system encoded in the form factor.

The definition of the charge density distribution for a spin-0 system was further scrutinized by Jaffe [11] in relationship to three characteristic length scales: the scale Δ set by the form factor slope, $\Delta^2 = 2dF'(q^2)|_{q^2=0}$ with d being the number of spatial dimensions, the characteristic size of the wave packet R and the Compton wavelength $1/m$. Using a Gaussian wave packet and an *approximate* expression for the charge distribution, Jaffe concluded that the interpretation of the Fourier transformed form factor as the intrinsic charge density is not valid for light hadrons and argued that local density distributions cannot even be defined independent of the form of the wave packet for systems with $\Delta \sim 1/m$.

In this Letter, we revisit the definition of the charge density for spin-0 systems. We closely follow the logic and conventions of Ref. [11], but we make no approximations to evaluate the charge density in a general wave packet state. Using spherically symmetric wave packets in the zero average momentum frame (ZAMF) of the system, we show that the charge density can be defined unambiguously for sharply localized packets. For wave packets with a sharp localization in momentum space, the ZAMF coincides with the rest frame of the system. We then generalize the definition to moving frames and show that in the infinite-momentum frame (IMF), the charge density turns into the well-known two-dimensional distribution in the transverse plane. We also discuss the relationship between the radial moments of the charge density and the form factor.

The charge density in the ZAMF.—Following Ref. [11], we consider, for the sake of definiteness, a spin-0 system. Notice, however, that spin plays no special role in the analysis below, which we expect to be applicable to any localizable quantum system. We assume that the system is an eigenstate of the charge operator $\hat{Q} = \int d^3r \hat{\rho}(\mathbf{r}, 0)$, $\hat{Q}|p\rangle = Q|p\rangle$, where $\hat{\rho}(\mathbf{r}, 0)$ is the electric charge density operator at $t = 0$ in the Heisenberg picture, and we take $Q = 1$ for definiteness. The momentum eigenstates $|p\rangle$ are normalized in the usual way,

$$\langle p'|p\rangle = 2E(2\pi)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p}), \tag{1}$$

with $p = (E, \mathbf{p})$, $E = \sqrt{m^2 + \mathbf{p}^2}$. Using translational invariance, the matrix elements of $\hat{\rho}(\mathbf{r}, 0)$ between momentum eigenstates of a spin-0 system can be written as

$$\langle p'|\hat{\rho}(\mathbf{r},0)|p\rangle = e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}(E+E')F(q^2), \qquad (2)$$

where $F(q^2)$ is the electric form factor and $q = p' - p$ denotes the momentum transfer.

Next, we define a normalizable Heisenberg-picture state of the system with the center-of-mass position $\mathbf{X}$ in terms of the wave packet

$$|\Phi, \mathbf{X}\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\phi(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{X}}|p\rangle, \qquad (3)$$

where the profile function $\phi(\mathbf{p})$ is required to satisfy

$$\int d^3p|\phi(\mathbf{p})|^2 = 1 \qquad (4)$$

in order to ensure the proper normalization of the wave packet. For later use, we define a dimensionless profile function $\tilde{\phi}$ via

$$\phi(\mathbf{p}) = R^{3/2}\tilde{\phi}(R\mathbf{p}), \qquad (5)$$

where R denotes the characteristic size of the wave packet with $R \to 0$ corresponding to a sharp localization. The charge density distribution in this state, defined as the matrix element of the electric charge density operator, has the form

$$\langle\Phi, \mathbf{X}|\hat{\rho}(\mathbf{r},0)|\Phi, \mathbf{X}\rangle = \int \frac{d^3p\, d^3p'}{(2\pi)^3\sqrt{4EE'}}(E+E')F(q^2) \times \phi^\star(\mathbf{p}')\phi(\mathbf{p})e^{i\mathbf{q}\cdot(\mathbf{X}-\mathbf{r})}, \qquad (6)$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ and $q^2 = (E'-E)^2 - \mathbf{q}^2$. Without loss of generality we choose $\mathbf{X} = 0$. Finally, introducing the total and relative momentum variables via $\mathbf{p} = \mathbf{P} - \mathbf{q}/2$ and $\mathbf{p}' = \mathbf{P} + \mathbf{q}/2$, the charge density is written as

$$\begin{aligned}\rho_\phi(\mathbf{r}) &\equiv \langle\Phi, \mathbf{0}|\hat{\rho}(\mathbf{r},0)|\Phi, \mathbf{0}\rangle \\ &= \int \frac{d^3P\, d^3q}{(2\pi)^3\sqrt{4EE'}}(E+E')F[(E-E')^2 - \mathbf{q}^2] \\ &\quad \times \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right)\phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right)e^{-i\mathbf{q}\cdot\mathbf{r}}, \qquad (7)\end{aligned}$$

where the energies are $E = \sqrt{m^2 + \mathbf{P}^2 - \mathbf{P}\cdot\mathbf{q} + \mathbf{q}^2/4}$ and $E' = \sqrt{m^2 + \mathbf{P}^2 + \mathbf{P}\cdot\mathbf{q} + \mathbf{q}^2/4}$.

The traditional ("naive") interpretation of the charge density in terms of the Fourier transform of the form factor in the Breit frame, $F(q^2) = F(-\mathbf{q}^2)$, emerges by first taking the static limit (i.e., $m \to \infty$) resulting in the replacement $E = E' = m$ in the integrand in Eq. (7),

$$\rho_{\phi,\text{naive}}(\mathbf{r}) = \int \frac{d^3P\, d^3q}{(2\pi)^3}\phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right)\phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) \times F(-\mathbf{q}^2)e^{-i\mathbf{q}\cdot\mathbf{r}}, \qquad (8)$$

and subsequently localizing the wave packet by taking the limit $R \to 0$ [11]. This can be done without specifying the functions $F(q^2)$ and $\phi(\mathbf{p})$ using the method of dimensional counting [15] or, alternatively, the strategy of regions [16]. For $F(q^2)$ decreasing at large q^2 faster than $1/q^2$, the only nonvanishing contribution to $\rho_{\phi,\text{naive}}(\mathbf{r})$ in the $R \to 0$ limit is obtained by substituting $\mathbf{P} = \tilde{\mathbf{P}}/R$, expanding the integrand in Eq. (8) in R around $R = 0$, and keeping the zeroth order term. The resulting naive charge density has the familiar form

$$\begin{aligned}\rho_\text{naive}(r) &= \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3}F(-\mathbf{q}^2)|\tilde{\phi}(\tilde{\mathbf{P}})|^2e^{-i\mathbf{q}\cdot\mathbf{r}} \\ &= \int \frac{d^3q}{(2\pi)^3}F(-\mathbf{q}^2)e^{-i\mathbf{q}\cdot\mathbf{r}}, \qquad (9)\end{aligned}$$

where in the second equality we made use of Eq. (4). Here and in what follows, $r \equiv |\mathbf{r}|$. We have dropped the subscript ϕ to indicate that the above expression is independent of the wave packet shape.

On the other hand, the method of dimensional counting allows one to take the $R \to 0$ limit in Eq. (7) without employing the static approximation. Following the same steps as before but for arbitrary m, we obtain

$$\rho_\phi(\mathbf{r}) = \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3}F\left[\frac{(\tilde{\mathbf{P}}\cdot\mathbf{q})^2}{\tilde{\mathbf{P}}^2} - \mathbf{q}^2\right]|\tilde{\phi}(\tilde{\mathbf{P}})|^2e^{-i\mathbf{q}\cdot\mathbf{r}}. \qquad (10)$$

The resulting density depends on the shape of the wave packet unless it is spherically symmetric. Since there is no preferred direction in the ZAMF of the system, we *define* the charge density distribution in the ZAMF by employing spherically symmetric wave packets with $\tilde{\phi}(\tilde{\mathbf{P}}) = \tilde{\phi}(|\tilde{\mathbf{P}}|)$ only. Then, using spherical coordinates to perform the integration over $\tilde{\mathbf{P}}$ in Eq. (10), we arrive at the final form of the charge density distribution in the ZAMF of a particle

$$\rho(r) = \int \frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{r}}\int_{-1}^{+1}d\alpha\frac{1}{2}F[(\alpha^2 - 1)\mathbf{q}^2]. \qquad (11)$$

While it is argued in Ref. [11] that the traditional result $\rho_\text{naive}(r)$ is valid for the hierarchy of scales $\Delta \gg 1/m$, comparing the approximate and exact expressions in Eqs. (9) and (11), respectively, shows that the accuracy of the static approximation leading to $\rho_\text{naive}(r)$ does not depend upon the particle mass m. It is also clear that the validity of Eq. (11), which provides an unambiguous relationship between the matrix element of the local charge density operator $\hat{\rho}(\mathbf{r},0)$ in a quantum system and the experimentally measurable form factor $F(q^2)$, does not depend on the relation between the intrinsic size of the system Δ and its Compton wavelength $1/m$ (in contrast to what is claimed in Ref. [11]).

Discussion.—A striking feature of the obtained result for $\rho(r)$ is its independence of the particle's mass. This implies that the traditional expression for the charge density, $\rho_{\rm naive}(r)$, does *not* emerge from $\rho(r)$ by taking the static limit: $\rho_{\rm naive}(r) \neq \lim_{m\to\infty}\rho(r)$. At first glance, this seems puzzling as one expects the conventional static result to be a better approximation for heavy systems like atoms or atomic nuclei [11,12]. The reason for this mismatch is the noncommutativity of the $R\to 0$ and $m\to\infty$ limits of $\rho_\phi(\mathbf{r})$ in Eq. (7), as implicitly shown in Figs. 1–3 of Ref. [11]. While the static limit and, more generally, the nonrelativistic approximation is perfectly valid when calculating the form factor in Eq. (2) provided $-q^2 \ll m^2$, it is violated in certain momentum regions when performing the integration in Eq. (7).

To have a simple example demonstrating the noncommutativity of the $m\to\infty$ and $R\to 0$ limits consider the wave packet in one spatial dimension with

$$\phi(p) = \sqrt{\frac{2R}{\pi}}\frac{1}{1+R^2p^2}, \tag{12}$$

and the form factor

$$F(q_0^2-q^2) = \frac{2}{2-\Delta^2(q_0^2-q^2)}, \tag{13}$$

so that $F(0)=1$ and $F'(0)=\Delta^2/2$. We calculate the second order moment of the charge distribution using the version of Eq. (7) in one spatial dimension,

$$\langle x^2\rangle_\phi = \int_{-\infty}^{+\infty} dx x^2 \int_{-\infty}^{+\infty}\frac{dPdq}{2\pi\sqrt{4EE'}}(E+E') \times F[(E-E')^2-kq^2]\phi\left(P-\frac{q}{2}\right)\phi^\star\left(P+\frac{q}{2}\right)e^{-iqx}. \tag{14}$$

For demonstration purposes, we have introduced a control parameter k to be set to $k=1$ in the final result. The integral in Eq. (14) can be easily calculated by writing the factors of x as derivatives acting on the exponential function. The resulting expression has the form

$$\langle x^2\rangle_\phi = k\Delta^2 - \frac{\Delta^2}{(1+mR)^2} + \frac{R^2}{2} - \frac{R}{4m(1+mR)^3}. \tag{15}$$

Taking the limit $R\to 0$ in Eq. (15) leads to

$$\langle x^2\rangle = (k-1)\Delta^2 = 0, \tag{16}$$

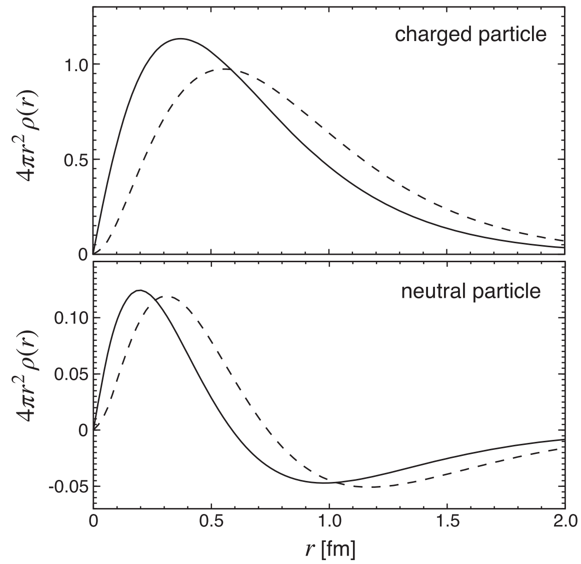

FIG. 1. Radial charge density distributions $4\pi r^2\rho(r)$ (solid lines) and $4\pi r^2\rho_{\rm naive}(r)$ (dashed lines) for a charged and a neutral spin-0 particle using the dipole and Galster-type parametrizations of the electric form factors, respectively.

which does not depend on the mass m. On the other hand, taking first the static limit $m\to\infty$ and subsequently the $R\to 0$ limit we obtain a different result

$$\langle x^2\rangle_{\rm naive} = k\Delta^2 = \Delta^2. \tag{17}$$

The method of dimensional counting reproduces exactly Eq. (16), while Eq. (17) is obtained by first taking the static limit in the integrand of Eq. (14).

We now turn to the interpretation of our result in Eq. (11). Clearly, the dependence of $\rho(r)$ on the angle-averaged form factor $\frac{1}{2}\int_{-1}^{+1} d\alpha F[(\alpha^2-1)\mathbf{q}^2]$ rather than the Breit frame expression $F(-\mathbf{q}^2)$ affects the radial profile of the charge density. To quantify the magnitude of this effect, we compare in Fig. 1 $\rho(r)$ and $\rho_{\rm naive}(r)$ for a charged and a neutral particle. For illustrative purposes we employ here simple parametrizations of the form factors, namely the dipole proton form factor $F_p(q^2)=G_D(q^2)=(1-q^2/\Lambda^2)^{-2}$ with $\Lambda^2=0.71\ {\rm GeV}^2$ for a charged scalar particle and the Galster-type parametrization of the neutron form factor from Ref. [17], $F_n(q^2)=A\tau/(1+B\tau)G_D(q^2)$ with $\tau=-q^2/(4m_p^2)$, $A=1.70$, and $B=3.30$, for a neutral scalar particle.

To gain further insights into the relationship between the charge density and the form factor it is instructive to rewrite Eq. (11) in coordinate independent form as

$$\rho(r) = \frac{1}{4\pi}\int d^2\hat{n}\rho_{\hat{\mathbf{n}}}(\mathbf{r}), \tag{18}$$

where $\hat{\mathbf{n}} \equiv \mathbf{n}/|\mathbf{n}|$ is a unit vector and

$$\rho_{\hat{\mathbf{n}}}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} F(-\mathbf{q}_\perp^2) e^{-i\mathbf{q}\cdot\mathbf{r}} = \rho_{\hat{\mathbf{n}}}(r_\parallel)\rho_{\hat{\mathbf{n}}}(r_\perp). \quad (19)$$

Here, $\mathbf{q}_\perp = \hat{\mathbf{n}} \times (\mathbf{q} \times \hat{\mathbf{n}})$, $\mathbf{r}_\perp = \hat{\mathbf{n}} \times (\mathbf{r} \times \hat{\mathbf{n}})$, $r_\parallel = \mathbf{r} \cdot \hat{\mathbf{n}}$, $r_\perp \equiv |\mathbf{r}_\perp|$, and the one- and two-dimensional densities in the $\hat{\mathbf{n}}$ and $\mathbf{r}_\perp$ directions are given by

$$\rho_{\hat{\mathbf{n}}}(r_\parallel) = \int \frac{dq_\parallel}{2\pi} e^{-iq_\parallel r_\parallel} = \delta(r_\parallel),$$
$$\rho_{\hat{\mathbf{n}}}(r_\perp) = \int \frac{d^2q_\perp}{(2\pi)^2} F(-\mathbf{q}_\perp^2) e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}, \quad (20)$$

with $q_\parallel = \mathbf{q} \cdot \hat{\mathbf{n}}$. These expressions establish a geometric interpretation of $\rho(r)$ to be discussed below and explain the squeezing of the radial charge density relative to the naive result as shown in Fig. 1.

To further elaborate on this point we compute radial moments of the charge distributions in $d \in \mathbb{N}$ spatial dimensions. We start with the inverse Fourier transform of Eq. (9)

$$F(-\mathbf{q}^2) = \int d^d r \rho_{\text{naive}}(r) e^{i\mathbf{q}\cdot\mathbf{r}}. \quad (21)$$

Taking the kth derivative of this expression at $-\mathbf{q}^2 = 0$, $F^{(k)}(0)$, we find for $\langle r^{2k}\rangle_{\text{naive}}^{(d)} \equiv \int d^d r \rho_{\text{naive}}(r) r^{2k}$:

$$\langle r^{2k}\rangle_{\text{naive}}^{(d)} = \frac{2^{2k}\Gamma(d/2+k)}{\Gamma(d/2)} F^{(k)}(0). \quad (22)$$

For $d = 3$, this reduces to the well-known expression

$$\langle r^{2k}\rangle_{\text{naive}} = \frac{(2k+1)!}{k!} F^{(k)}(0). \quad (23)$$

On the other hand, using Eqs. (18)–(20) generalized to d dimensions and noting that $\langle r^{2k}\rangle_{\hat{\mathbf{n}}}^{(d)} \equiv \int d^d r \rho_{\hat{\mathbf{n}}}(\mathbf{r}) r^{2k}$ does not depend of $\hat{\mathbf{n}}$, we obtain for $d \geq 2$

$$\langle r^{2k}\rangle^{(d)} = \langle r^{2k}\rangle_{\text{naive}}^{(d-1)}, \quad (24)$$

so that in $d = 3$ spatial dimensions,

$$\langle r^{2k}\rangle = 2^{2k} k! F^{(k)}(0), \quad (25)$$

see Ref. [18] for a related discussion. Notice further that in one spatial dimension, $\rho(r) = \delta(r)$ independently of the form factor. This explains the vanishing result for the second moment in the considered one-dimensional example, see Eq. (16). The vanishing argument of the form factor in the one-dimensional variant of Eq. (10) implies that for $d = 1$, $\rho(r)$ probes only the total charge $F(0)$ and not the internal structure of the system encoded in $F(q^2)$.

The charge density in moving frames.—While the static approximation $\rho_{\text{naive}}(r)$ does not depend on the frame, the expressions for $\rho(r)$ in Eqs. (11) and (18) are valid in the ZAMF of the system. It is straightforward to generalize these results to a boosted frame.

We start with the general expression for $\rho_\phi(\mathbf{r})$ in Eq. (7) and replace $\phi(\mathbf{p})$ with $\phi_\mathbf{v}(\mathbf{p})$, where $\mathbf{v}$ denotes the boost velocity. Differently to $\phi(\mathbf{p})$, we cannot regard the function $\phi_\mathbf{v}(\mathbf{p})$ to be spherically symmetric. Thus, we need to express $\phi_\mathbf{v}(\mathbf{p})$ in terms of the quantity $\phi(\mathbf{p})$ defined in the ZAMF in order to obtain a wave packet independent definition for the charge density in the $R \to 0$ limit. Using Eq. (3) and the Lorentz transformation properties of the momentum eigenstates $|p\rangle$, one finds [19]

$$\phi_\mathbf{v}(\mathbf{p}) = \sqrt{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{p}}{E}\right)} \phi[\mathbf{p}_\perp + \gamma(\mathbf{p}_\parallel - \mathbf{v}E)], \quad (26)$$

where $\gamma = (1 - v^2)^{-1/2}$, $\mathbf{p}_\parallel = (\mathbf{p}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$, $\mathbf{p}_\perp = \mathbf{p} - \mathbf{p}_\parallel$, and $E = \sqrt{m^2 + \mathbf{p}^2}$. We note in passing that Eq. (26) ensures the invariance of the normalization of the wave packet [19]. Then, following the same steps as in the case of the ZAMF and using the method of dimensional counting to evaluate the $R \to 0$ limit we arrive at

$$\rho_{\phi,\mathbf{v}}(\mathbf{r}) = \int \frac{d^3\tilde{P} d^3q}{(2\pi)^3} \frac{\gamma(\tilde{P} - \mathbf{v}\cdot\tilde{\mathbf{P}})}{\tilde{P}} F\left[\frac{(\tilde{\mathbf{P}}\cdot\mathbf{q})^2}{\tilde{\mathbf{P}}^2} - \mathbf{q}^2\right] \times |\tilde{\phi}[\tilde{\mathbf{P}}_\perp + \gamma(\tilde{\mathbf{P}}_\parallel - \mathbf{v}\tilde{P})]|^2 e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad (27)$$

where $\tilde{P} \equiv |\tilde{\mathbf{P}}|$. We now change the integration variable $\tilde{\mathbf{P}} \to \tilde{\mathbf{P}}' = \tilde{\mathbf{P}}_\perp + \gamma(\tilde{\mathbf{P}}_\parallel - \mathbf{v}\tilde{P})$. Using the relations $\tilde{\mathbf{P}}_\parallel = \gamma(\tilde{\mathbf{P}}'_\parallel + \mathbf{v}\tilde{P}')$ and $\tilde{P} = \gamma(\tilde{P}' + v\tilde{P}'_\parallel)$, it is easy to verify that the Jacobian of the change of variables $\tilde{\mathbf{P}} \to \tilde{\mathbf{P}}'$ cancels the first factor in the integrand in Eq. (27), yielding

$$\rho_{\phi,\mathbf{v}}(\mathbf{r}) = \int \frac{d^3\tilde{P}' d^3q}{(2\pi)^3} |\tilde{\phi}(\tilde{\mathbf{P}}')|^2 e^{-i\mathbf{q}\cdot\mathbf{r}} \times F\left\{\frac{[\tilde{\mathbf{P}}'_\perp\cdot\mathbf{q}_\perp + \gamma(\tilde{\mathbf{P}}'_\parallel + \mathbf{v}\tilde{P}')\cdot\mathbf{q}_\parallel]^2}{\gamma^2(\tilde{P}' + v\tilde{P}'_\parallel)^2} - \mathbf{q}^2\right\}. \quad (28)$$

Using Eq. (4) and the spherical symmetry of $\tilde{\phi}(\tilde{\mathbf{P}}')$, the integration over $\tilde{P}'$ becomes trivial. The remaining angular integration over $\hat{\tilde{\mathbf{P}}}'$ can be done in spherical coordinates. We align the z and x axes along the $\mathbf{v}$ and $\mathbf{q}_\perp$ directions, respectively, and denote $\eta = \cos\theta$. Our final result then reads

$$\rho_\mathbf{v}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} \bar{F}(q_\parallel, q_\perp) e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad (29)$$

with $q_\parallel \equiv \hat{\mathbf{v}}\cdot\mathbf{q}$, $q_\perp \equiv |\mathbf{q}_\perp|$ and

$$\bar{F}(q_\parallel, q_\perp) = \frac{1}{4\pi}\int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \times F\left\{\frac{[\sqrt{1-\eta^2}\cos\phi q_\perp + \gamma(\eta+v)q_\parallel]^2}{\gamma^2(1+v\eta)^2} - \mathbf{q}^2\right\}. \quad (30)$$

In the IMF with $v \to 1$ and $\gamma \to \infty$, the charge density turns into the usual two-dimensional distribution in the transverse plane, $\rho_{\rm IMF}(\mathbf{r}) = \delta(r_\parallel)\rho_{\rm IMF}(r_\perp)$ with

$$\rho_{\rm IMF}(r_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2} F(-\mathbf{q}_\perp^2) e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}. \quad (31)$$

One can also verify that Eq. (29) reduces to the ZAMF expression in Eq. (11) in the limit $v \to 0$, albeit this relationship appears somewhat obscured.

Again, it is instructive to rewrite Eqs. (29) and (30) in a coordinate independent form similar to the ZAMF expressions in Eqs. (18)–(20). We introduce a unit vector $\hat{\mathbf{m}} \equiv \hat{\tilde{\mathbf{P}}}'$ and define a vector valued function

$$\mathbf{n}(\mathbf{v}, \hat{\mathbf{m}}) = \hat{\mathbf{v}} \times (\hat{\mathbf{m}} \times \hat{\mathbf{v}}) + \gamma(\hat{\mathbf{m}}\cdot\hat{\mathbf{v}} + v)\hat{\mathbf{v}}. \quad (32)$$

Then, the charge density $\rho_{\mathbf{v}}(\mathbf{r})$ can be written as

$$\rho_{\mathbf{v}}(\mathbf{r}) = \frac{1}{4\pi}\int d^2\hat{m}\,\rho_{\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}})}(\mathbf{r}), \quad (33)$$

where $\rho_{\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}})}(\mathbf{r}) \equiv \rho_{\hat{\mathbf{n}}}(\mathbf{r})$ is defined in Eqs. (19) and (20). In this form, both extreme limits for the boosting velocity become particularly transparent by using the relations $\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}}) \xrightarrow{v\to 0} \hat{\mathbf{m}}$ and $\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}}) \xrightarrow{v\to 1} \hat{\mathbf{v}}$, leading evidently to Eqs. (18) and (31), respectively.

The interpretation of the obtained results follows directly from the corresponding coordinate independent expressions. In the ZAMF, Eqs. (18)–(20) show that $\rho(r)$ is given by a continuous (isotropic) superposition of the two-dimensional "images" of the system, $\rho_{\rm IMF}(\mathbf{r})$, made in all possible IMFs. It is intuitively clear that the full image of a three-dimensional object can be reconstructed by putting together all possible two-dimensional projections. In moving frames, the averaging over the infinite momentum directions in Eq. (33) becomes anisotropic, reflecting a preferred direction set by the velocity $\mathbf{v}$.

This geometric picture also provides an interpretation of the peculiar result $\rho(r) = \delta(r)$ in one spatial dimension. The $(d-1)$-dimensional projections simply do not exist for $d = 1$, and $\rho(r)$ thus loses the information about the structure of the object. Nevertheless, the formal result $\rho(r) = \delta(r)$ for $d = 1$ constitutes a consistency check of our work. Indeed, given that the condition for the profile function $\phi(\mathbf{p})$ to be spherically symmetric drops out for $d = 1$, the same definition of the charge density should hold in the ZAMF and moving frames, making $\rho_v(r)$ frame independent. This then leads to $\rho(r) = \delta(r)$ as the only possibility compatible with the obvious IMF result.

Last but not least, we emphasize that radial moments of the charge distribution are, in fact, frame independent, i.e., $\langle r^{2k}\rangle_{\mathbf{v}} = \langle r^{2k}\rangle$, in spite of $\rho_{\mathbf{v}}(\mathbf{r})$ being not spherically symmetric for $v \neq 0$. This remarkable feature follows from Eq. (33) by noting that $\int d^3r\,\rho_{\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}})}(\mathbf{r})r^{2k}$ does not depend on $\mathbf{v}$ and $\hat{\mathbf{m}}$. It can also be verified by showing that radial moments of $\rho_\phi(\mathbf{r})$ in Eq. (10) do not depend on $\tilde{\phi}(\tilde{\mathbf{P}})$ even if this function is not spherically symmetric.

Summary and conclusions.—In summary, we introduced an unambiguous definition of a spatial distribution of the expectation values of local operators in spin-0 systems independent of the specific form of the wave packet in which the state was prepared. Our definition also applies to systems whose intrinsic size is comparable or even smaller than the Compton wavelength. We found remarkably simple relationships between the electric form factor and the charge density in the ZAMF and moving frames, thereby reproducing the well-known result in the IMF. We have also demonstrated that radial moments of the charge distribution are frame independent.

Our results suggest an unconventional interpretation of the form factors in terms of the charge density. In particular, Eq. (11) implies that the second moment of $\rho(r)$ is related to the form factor slope via $\langle r^2\rangle = 4F'(0)$ in contrast to the usual relationship $\langle r^2\rangle_{\rm naive} = 6F'(0)$ motivated by the Breit frame distribution $\rho_{\rm naive}(r)$. Our results show that the approximation $\rho_{\rm naive}(r)$ does not emerge in the static limit of the exact expression for $\rho(r)$, and its accuracy is independent of the particle's mass.

We note that for heavy systems with $(m\Delta)^{-1} \equiv \epsilon \ll 1$, one may alternatively attempt to define $\rho(\mathbf{r})$ using wave packets with $\epsilon \ll (mR)^{-1} \ll 1$ as suggested in [11], by choosing, e.g., $(mR)^{-1} \sim \mathcal{O}(\epsilon^{1/2})$. While this leads to an unambiguous definition of the charge density in the static limit $\epsilon \to 0$ with $\rho(\mathbf{r}) \to \rho_{\rm naive}(r)$, corrections beyond this limit are wave packet dependent.

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