

Revisiting a Low-Dimensional Model with Short Range Interactions and Mean Field Critical Behavior

Peter Grassberger¹

¹*JSC, FZ Jülich, D-52425 Jülich, Germany*

(Dated: October 28, 2021)

In all local low-dimensional models, scaling at critical points deviates from mean field behavior – with one possible exception. This exceptional model with “ordinary” behavior is an inherently non-equilibrium model studied some time ago by H.-M. Bröker and myself. In simulations, its 2-dimensional version suggested that two critical exponents were mean-field, while a third one showed very small deviations. Moreover, the numerics agreed almost perfectly with an explicit mean field model. In the present paper we present simulations with much higher statistics, both for 2d and 3d. In both cases we find that the deviations of all critical exponents from their mean field values are non-leading corrections, and that the scaling is *precisely* of mean field type. As in the original paper, we propose that the mechanism for this is “confusion”, a strong randomization of the phases of feed-backs that can occur in non-equilibrium systems.

Non-linear low-dimensional stochastic systems with short range interactions tend to show anomalously large fluctuations. These fluctuations then lead to “anomalous” scaling laws which deviate from their mean field behavior. This applies to virtually all sorts of systems: To rough surfaces (e.g. Kardar-Parisi-Zhang [1] and quenched Edwards-Wilkinson [2] models), to self-organized critical models like the Bak-Tang-Wiesenfeld [3] and Manna [4] sand pile models and the Bak-Sneppen evolution model [5], to heat conduction in 1-d systems such as the Fermi-Pasta Ulam [6] and alternating mass hard particle [7] systems, and – maybe most famously – to second order phase transitions as e.g. in Ising, XY, or Heisenberg models [8], in percolation [9], and in self-avoiding walks [10].

The possibility of “normal”, i.e. mean field type behavior has been much discussed in 1-d heat conduction, where “normal” behavior would correspond to the validity of Fourier’s law [11–17], but not in any of the other types of phenomena. It is true that in some cases even deviations from power law scaling have been observed (as in the Drossel-Schwabl forest fire model [18, 19]), but normal (i.e. mean-field) scaling at critical points were *never* reported – with one single exception. This exception is an old paper by H.-M. Bröker and myself [20]. This paper was cited only 5 times within 26 years (according to Google Scholar), which illustrates most clearly that the concept of a low-dimensional local model with mean field type scaling at a critical point was considered as completely outlandish by the community.

The model studied in [20] was a modification of Manna’s sandpile model (for a similar model, see [21]), with enhanced stochasticity and with non-conservation of ‘sand’. The latter is controlled by an explicit control parameter, which changes it from being self-organized critical into a model with a standard non-equilibrium second-order (continuum) phase transition. As in most such models, its critical behavior is characterized by three independent exponents. Two of these were found in [20]

to be mean field like, while the third showed very small deviations. Moreover, not only the exponents but also the scaling functions were extremely close to those of an explicit mean field model, namely to a version of the model on a Bethe tree. We should add that this model should, by standard arguments [21]), be in the universality class of the fixed-energy Manna sand pile.

The closeness to mean field suggested of course that the deviations from it could be finite-size corrections, but simulations at that time were unable to decide this question. We thus decided to revisit the problem and to perform much larger simulations with modern hardware. The results presented below are quite unambiguous: It seems that all deviations are indeed due to finite-size corrections, both for the 2-dimensional version of the model studied in [20] and for its generalization to 3 dimensions.

This model is defined on a d -dimensional hypercubic lattice (generalizations to other types of lattices are obvious), and time is discrete. At each lattice site we have a “spin” z_i , which can take any non-negative integer value, but only the values $z_i = 0$ and $z_i = 1$ are “stable”. If z_i becomes > 1 during the evolution, it “topples”. The toppling rule is

$$z_i \rightarrow z_i - 2 \quad (1)$$

for the site which topples, and

$$z_j \rightarrow \begin{cases} z_j + 1 & : \text{ with probability } p \\ z_j & : \text{ with probability } 1 - p \end{cases} \quad (2)$$

for each of its $2d$ neighbors, with $0 \leq p \leq 1/d$. Notice that each neighbor j has the same chance p to get its spin increased, *independently* of what happens at the other neighbors. Thus the sum $\sum_i z_i$ fluctuates (during each toppling, it can change by any value between -2 and $2d - 2$), but in the average each toppling event causes $\sum_i z_i$ to decrease by $2dp - 2$. The critical point is exactly where this vanishes, $p_c = 1/d$. For later use we define $\epsilon = 1/d - p$, and $\varrho = \langle z \rangle$ [22].

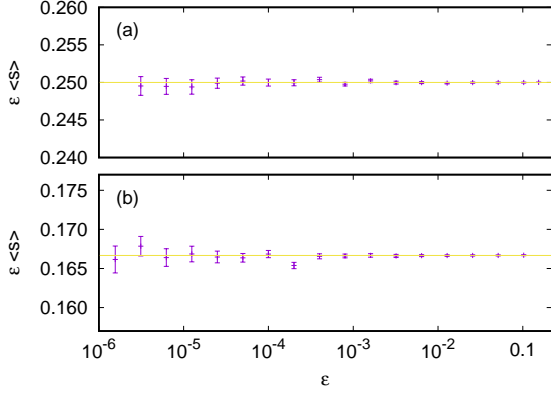


FIG. 1. Average avalanche sizes (number of topplings per event) plotted against ϵ , for (a) $d = 2$ (top panel) and (b) $d = 3$ (bottom panel). The horizontal lines indicate the theoretical predictions.

As in the sandpile model, the dynamics actually consists of the following rules:

- (i) We start with a configuration where all sites are stable.
- (ii) A site i is chosen randomly, and z_i is increased by one unit. In the following, we call this an “event”.
- (iii) If at least one site is unstable, the above toppling rule is applied immediately (i.e. without increasing the time counter) and simultaneously at all unstable sites [24]. After this, t is increased by 1 unit. If some z ’s are still ≥ 4 so that they have to topple again, then all these topplings are also done simultaneously. After each such round of topplings, t is again increased by 1. This is repeated until all sites are stable again, after which rule (ii) is applied again.

As we have already said, this is very similar to the model of [21]. The main difference is that we increase the neighboring ‘spins’ independently during a toppling, which adds to the randomness of the process. We believe that it is this enhanced degree of stochasticity which is responsible for the very special features of the model.

Although we could also run the model for $p \geq 1/2d$, if we would use open boundary conditions (as in the original versions of sandpile models), we show here only results for periodic (more precisely, helical) boundary conditions. In all simulations, sufficiently long transients were discarded so that all measurements are for the statistically stationary state. The number of events were for each pair of values (d, ϵ) more than 6×10^8 .

Since $\sum_i z_i$ decreases in average by $2d\epsilon$ during each toppling, while it first is increased by 1 in each event, the average number of topplings per event is just $1/(2d\epsilon)$. Verifying this (see Fig. 1) presents thus a stringent test that stationarity had been reached.

This does not, however, test against finite size corrections. But no such corrections whatsoever were seen, if we changed the simulation volume between 32×32 and

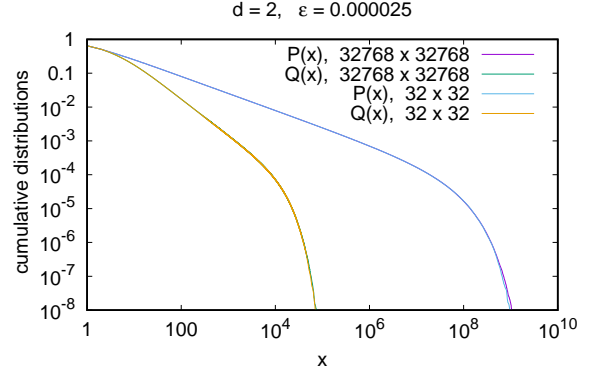


FIG. 2. Cumulative distributions of the number of topplings per event, ($P(x)$, upper curve) and of their durations ($Q(x)$, lower curve), for $d = 2$ and $\epsilon = 0.000025$. Even though only two curves seem to be visible, four curves are actually shown: Each distribution is shown for $L = 32$ and for $L = 32768$, where the simulation boxes have sizes $L \times L$.

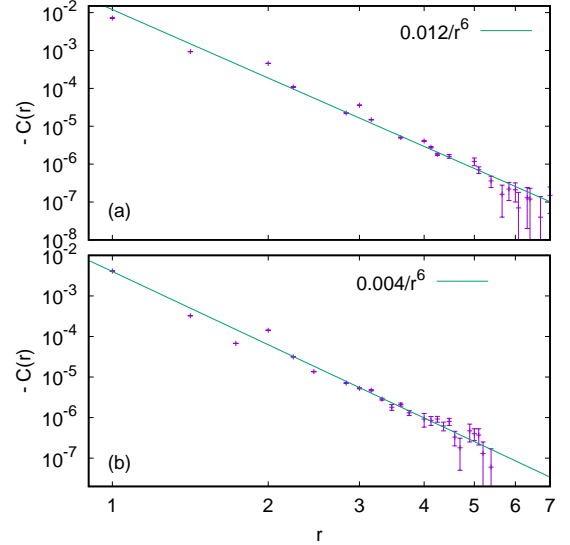


FIG. 3. Spatial correlations between ‘spin’ values (or densities, in a sandpile terminology) distances r apart. Since all correlations for all $r > 0$ are negative, we show a log-log plot for $-C(r)$ vs. r . The straight lines give rough power law fits, but notice that deviations from these power laws are significant and rather erratic. As in Fig. 1 and in the following figures, panel (a) is for $d = 2$, while (b) is for $d = 3$.

32768×32768 in 2 dimensions, and between 16^3 and 512^3 in $d = 3$ (see Fig. 2). This complete absence of visible finite size effects is very surprising (for many sandpile models, finite size effects are huge, see e.g. [2]). It allowed us to use rather modest lattice sizes: In $d = 2$, most simulations were done with $L = 256$ or 512 , and in $d = 3$ we used mostly $L = 128$ and 256 – although the largest simulated avalanches were huge in both cases, and had $\approx 10^{11}$ topplings. It is easily explained by the

smallness and short ranges of correlations. As seen from Fig. 3, these correlations are negative both in $d = 2$ and $d = 3$, and

$$c(i-j) \equiv \langle z_i z_j \rangle - \varrho^2 \sim ||i-j||^{-6} \quad (3)$$

in both dimensions, which is an even faster decay than in the Bak-Tang-Wiesenfeld sandpile model (where $c(i-j) \sim ||i-j||^{-4}$ [26]). Notice that the absolute values of the correlations are much smaller in $d = 3$ than in $d = 2$. This is expected if the behavior is close to mean field, because deviations from it should decrease with d .

Average stationary densities ϱ versus ϵ are shown in Fig. 4. We also show there predictions from a mean field model, following [20]. In this model, a site ‘remembers’ for the present time step that it had toppled, but ‘forgets’ it thereafter. Thus a site that had not toppled during the present avalanche has density ϱ . A toppling site which is not at the root of an avalanche has thus $2d-1$ neighbors with density ϱ , while its ‘father’ (the site which made it topple) has a different density ϱ' . In the simplest version, we also neglect the possibility that the father might have changed after this toppling, which implies $\varrho' = 0$, in which case the problem is essentially that of percolation on a Bethe lattice with coordination number $2d$ [9]. But we can also allow values $\varrho' \neq 0$, in which case the model can still be solved exactly [20]. In any case we expect $\varrho' \ll \varrho$. Let us denote by $a = p\varrho$ and $a' = p\varrho'$ the probabilities that a toppling (non-root) site will make its neighbors topple, while the very first toppling of an avalanche will make all its neighbors topple with probability a . The average number of topplings during an event (whether it triggered an avalanche or not) is [20]

$$\langle s(p) \rangle = \varrho [1 + 2da \sum_{i=0}^{\infty} (2da + a' - a)^i] = \varrho \frac{1 + a - a'}{1 - (2d-1)a - a'}. \quad (4)$$

Since we know that $\langle s(p) \rangle = 1/(2d\epsilon)$, we obtain

$$\varrho = \frac{1}{2} + \frac{2a(a-a')}{1 + (2d-1)a - a'}, \quad \epsilon = 1/d - a/\varrho. \quad (5)$$

This allows us to obtain ϱ as a function of ϵ for any fixed ratio a'/a . From Fig. 4 we see that very good fits are obtained in both dimensions with $a' = 0$. While this is indeed the best fit in $d = 3$, an even better fit is obtained in $d = 2$ with $a' \sim a^3$.

Finally, we show in Figs. 5 to 8 the (cumulative) distributions $P(s)$ of the number s of topplings per event and $Q(t)$ of the time durations t . We could make detailed comparison with the mean field model as in [20], but we prefer to just show that the scaling predictions

$$P(s) = \frac{1}{\sqrt{s}} \Psi(\epsilon^2 s) \quad (6)$$

and

$$Q(t) = \frac{1}{t} \Phi(\epsilon t) \quad (7)$$

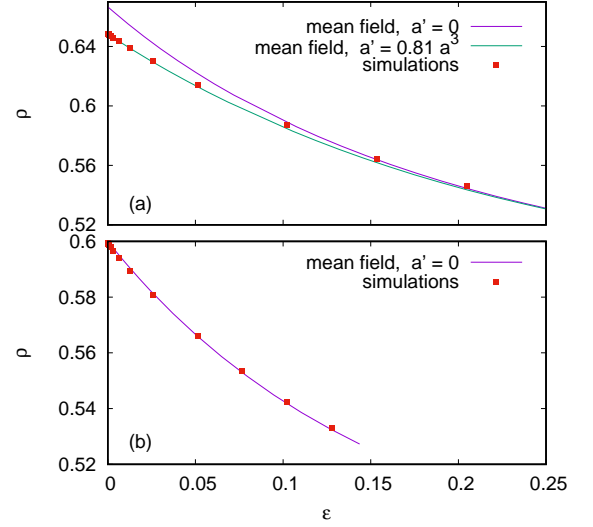


FIG. 4. Stationary densities (average spins), measured between events. Statistical errors are much smaller than the data points. For $d = 2$ (panel(a)) we show also two mean field predictions, one (the upper smooth curve) for $a = 0$, the other for $a' = 0.81a^3$. For $d = 3$ (panel (b)) we only show the mean field curve for $a' = 0$, as this gives already a nearly perfect fit.

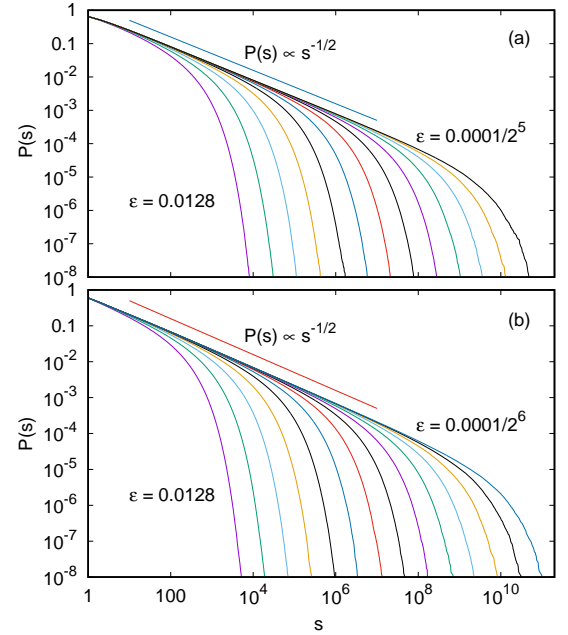


FIG. 5. Log-log plots of the cumulative distributions of the number of topplings per event. Here and in the following plots, the curves are for $\epsilon = \epsilon_{\min}, 2\epsilon_{\min}, 4\epsilon_{\min}, \dots, \epsilon_{\max}$. Statistical errors are significant only in the far right tails, as indicated by the (very small) fluctuations. The similarity between the two sets of curves reflects the fact that both are essentially those for the mean field model.

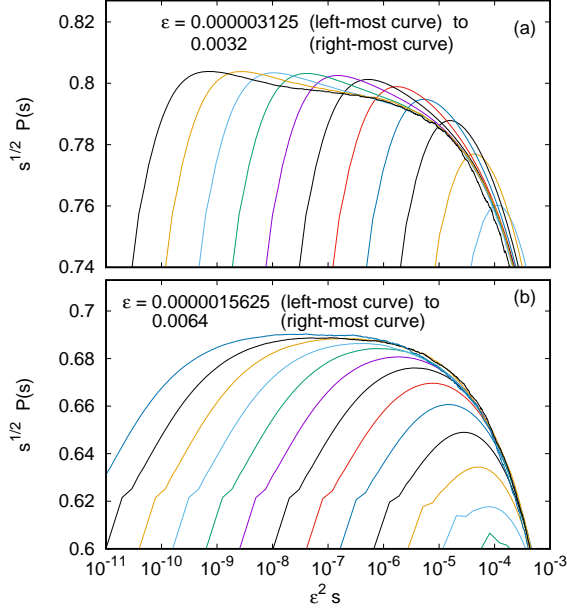


FIG. 6. Same data as in Fig. 5, but plotted as $\sqrt{s}P(s)$ against $\epsilon^2 s$. According to Eq. (6), these curves should tend towards straight horizontal curves for $\epsilon \rightarrow 0$. To see more clearly whether this is true or not, we show the data on a strongly blown up (non-logarithmic) y-scale.

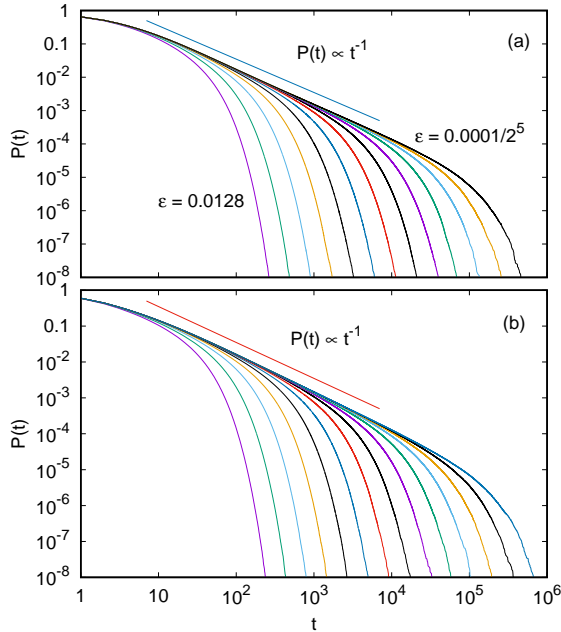


FIG. 7. Log-log plots of the cumulative distributions of the durations t of avalanches. Again, statistical errors are significant only in the far right tails, and the similarity between the two sets of curves reflects again the fact that both are essentially those for the mean field model.

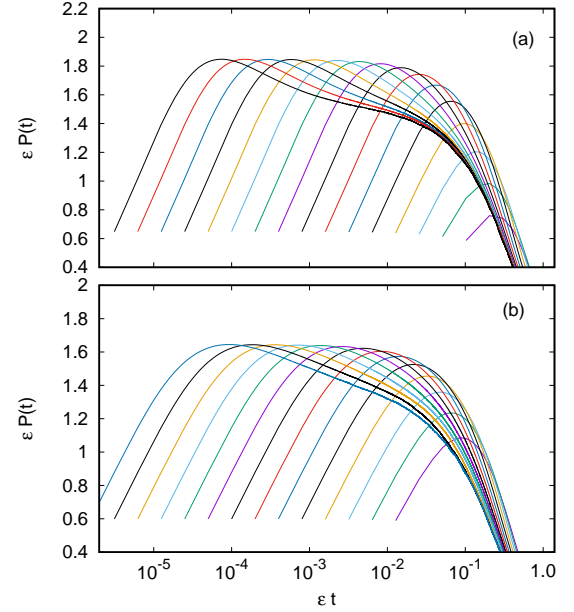


FIG. 8. Same data as in Fig. 7, but plotted as $tQ(t)$ against ϵt . According to Eq. (7), these curves should tend towards straight horizontal curves for $\epsilon \rightarrow 0$. To see more clearly whether this is true or not, we show the data on a strongly blown up (non-logarithmic) y-scale.

are excellently fulfilled in the asymptotic region $\epsilon \rightarrow 0$. For both we show (in Figs. 5 and 7) first the unmodified distributions in their entire ranges, but since this is not really significant we then plot (in Figs. 6 and 8) blow-ups of the scaling parts for $\sqrt{s}P(s)$ versus $\epsilon^2 n$ and for $tQ(t)$ versus ϵt . In all four plots (for $d = 2$ and $d = 3$, and for P and Q) we see clear deviations from scaling (i.e., none of the curves are horizontal in the central regions), but in all four cases these seem clearly to disappear for $\epsilon \rightarrow 0$.

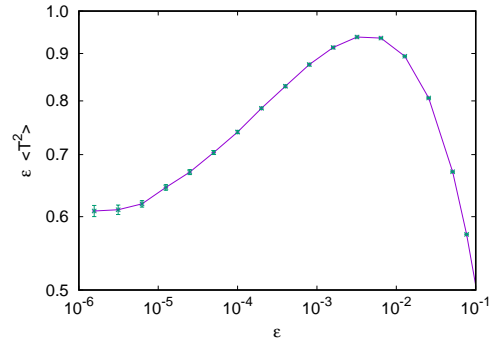


FIG. 9. Log-linear plot of the average squared avalanche durations in $d = 3$ versus ϵ . For increased significance we show $\epsilon \langle t^2 \rangle$ instead of $\langle t^2 \rangle$ itself. We see large corrections to scaling, but for $\epsilon \rightarrow 0$ the curve becomes horizontal as predicted by Eq. (7).

In Fig. 8b we also see quite a substantial violation of

scaling for large values of t (curves do not collapse there), but a closer inspection shows that this also disappears for $\epsilon \rightarrow 0$. This is also seen more clearly from Fig. 9, where we show $\langle t^2 \rangle$ for $d = 3$. Eq.(7) implies that $\langle t^2 \rangle \propto 1/\epsilon$ or

$$\epsilon \langle t^2 \rangle = \text{const}, \quad (8)$$

but we see that this only becomes true for extremely small values of ϵ . Basically the same happens in $d = 2$, although the scaling violations there are much smaller (data not shown). It is precisely the scaling violations seen in Figs. 6,8, and 9 which were at the basis of claims in [20] that mean field scaling is not exact, because at that time we were unable to simulate at as small values of ϵ as in the present paper.

In summary, we have shown that the critical scaling in the model of [20] is indeed precisely that of its mean field version, both for $d = 2$ and $d = 3$. To our knowledge this is the first and only model in low dimensions and with short-range interactions where this was ever observed. Indeed, mean field scaling usually does not occur in such models, if they show detailed balance, i.e. if they describe equilibrium phenomena, since feedback loops there tend to have a definite sign. Take, e.g., the Ising model. Loops contribute obviously with a positive sign in the ferromagnetic Ising model, but also in the antiferromagnetic one on bipartite lattices. The same is true for other models like self avoiding walks, the Heisenberg model, and percolation. We are not aware of a proof that the same holds for all equilibrium models, thus a search for mean field behavior in equilibrium critical phenomena might be worth while. In the present (non-equilibrium) model, these arguments do not apply because multiple topplings lead to ‘spin’ changes of either signs, and thus to cancellations in the contribution of loops. As in [20], we propose to call this phenomenon “confusion”. It can be viewed as the first known instance where the basic concept of the random phase approximation [27, 28] becomes exact, and even that only at a critical point.

[1] M. Kardar, and G. Parisi, “Dynamic scaling of growing interfaces”, Phys. Rev. Lett. **56**, 889 (1986).
[2] P. Grassberger, D. Dhar, and P.K. Mohanty, “Oslo model, hyperuniformity, and the quenched Edwards-Wilkinson model”, Phys. Rev. E **67**, 030301 (2016).
[3] P. Bak, C. Tang and K. Wiesenfeld, “Self-organized criticality: An explanation of the $1/f$ noise”, Phys. Rev. Lett. **59** (1987) 381.
[4] S.S. Manna, “Two-state model of self-organized criticality”, J. Phys. A: Math. Gen. **24**, L363 (1991).
[5] P. Bak and K. Sneppen, “Punctuated equilibrium and criticality in a simple model of evolution”, Phys. Rev. Lett. **71**, 4083 (1993).
[6] G.P. Berman and F.M. Izrailev, “The Fermi–Pasta–Ulam problem: fifty years of progress”, Chaos **15**, 015104 (2005).

[7] P. Grassberger, W. Nadler, and L. Yang, “Heat conduction and entropy production in a one-dimensional hard-particle gas”, Phys. Rev. Lett. **89**, 180601 (2002).
[8] D. Amit and V. Martin-Mayor, *Field theory, the renormalization group, and critical phenomena: graphs to computers* (World Scientific, New Jersey 2005).
[9] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London 1994).
[10] P.G. de Gennes, *Scaling concepts in polymer physics* (Cornell Univ. Press, Cornell, Ithaca, 1979).
[11] G. Casati, “Energy transport and the Fourier heat law in classical systems”, Found. Phys. **16**, 51 (1986).
[12] P.L. Garrido, P.I. Hurtado, and B. Nadrowski, “Simple one-dimensional model of heat conduction which obeys fourier’s law”, Phys. Rev. Lett. **86**, 5486 (2001).
[13] O. Gendelman and A. Savin, “Normal heat conductivity of the one-dimensional lattice with periodic potential of nearest-neighbor interaction”, Phys. Rev. Lett. **84**, 2381 (2000).
[14] O. Gendelman and A. Savin, “Normal heat conductivity in chains capable of dissociation”, EPL **34004** (2014).
[15] L. Yang and P. Grassberger, “Are there really phase transitions in 1-d heat conduction models?”, arXiv:cond-mat/0306173 (2003).
[16] G. Casati and B. Li, “Heat conduction in one dimensional systems: Fourier law, chaos, and heat control”, in: Khanna F. & Matrasulov D. (eds.) “Non-Linear Dynamics and Fundamental Interactions”, NATO Science Series II: Mathematics, Physics and Chemistry, vol 213. Springer, Dordrecht, 2006).
[17] S. Lepri, R. Livi, and A. Politi, “Too close to integrable: Crossover from normal to anomalous heat diffusion”, Phys. Rev. Lett. **125**, 040604 (2020).
[18] G. Pruessner and H.J. Jensen, “Broken scaling in the forest-fire model”, Phys. Rev. E **65**, 056707 (2002),
[19] P. Grassberger, “Critical behavior of the Drossel-Schwabl forest fire model”, New J. Phys. **4**, 17 (2002).
[20] H.-M. Bröker and P. Grassberger, “Mean-Field Behavior in a Local Low-Dimensional Model”, EPL **30**, 319 (1995).
[21] S.S. Manna, L.B. Kiss and J. Kertesz, “Cascades and self-organized criticality” J. Stat. Phys. **61** (1990) 923.
[22] This model had actually been mentioned first as model #5 in the appendix of [23], because its depth-first (recursive) simulation leads to an algorithm very close to percolation, to self-avoiding walks, and to the Wolff algorithm for the Ising model.
[23] P. Grassberger, “Recursive sampling of random walks: self-avoiding walks in disordered media” J. Phys. A **26**, 1023 (1993).
[24] Since the model is Abelian in the sense of [25], the order in which these topplings are performed is irrelevant.
[25] D. Dhar, “Self-organized critical state of sandpile automaton models”, Phys. Rev. **64**, 1613 (1990).
[26] S.N. Majumdar and D. Dhar, “Equivalence between the Abelian sandpile model and the $q \rightarrow 0$ limit of the Potts model”, Physica A **185**, 129 (1992).
[27] D. Bohm and D. Pines, “A Collective Description of Electron Interactions, I – III”, Phys. Rev. **82**, 625 (1951); Phys. Rev. **85**, 338 (1952); Phys. Rev. **92**, 609(1953).
[28] M. Gell-Mann and K. Brueckner, “Correlation Energy of an Electron Gas at High Density”, Phys. Rev. **106**, 364 (1957).