Definition of electromagnetic local spatial densities for composite spin-1/2 systems

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(Received 8 June 2022; accepted 1 September 2022; published 23 September 2022)

A new definition of the electromagnetic spatial densities for a spin-1/2 system is proposed and worked out in the zero average momentum frame and in moving frames. The obtained results are compared with the traditional definition of the densities in terms of the three-dimensional Fourier transforms of the electromagnetic form factors in the Breit frame.

DOI: 10.1103/PhysRevD.106.056019

I. INTRODUCTION

The three-dimensional Fourier transform of the charge form factor in the Breit frame is often interpreted as the electric charge density of the corresponding hadron, owing to the seminal papers on electron-proton scattering by Hofstadter, Sachs, and others in the 1960s [1–3]. Similar interpretations have also been proposed for the Fourier transforms of the gravitational form factors and for various local distributions [4–6].

Despite common belief, the identification of spatial density distributions with the Fourier transform of the corresponding form factors in the Breit frame suffers from conceptual issues as repeatedly pointed out in the literature [7–13]. In Ref. [11], it was shown on the example of a spin-0 system that the above-mentioned traditional expression for the charge density in terms of the Breit frame distribution follows only in the static limit of an infinitely heavy particle. On the other hand, doubt has been raised that local density distributions can even be defined unambiguously, i.e., independently of the form of the wave packet, for systems whose Compton wavelength is of the order of or larger than the charge radius (defined via the derivative of the form factor at zero momentum transfer) [11].

The issue of a proper definition of the spatial distributions of matrix elements of local operators has attracted much attention in the last few years. For example, the light-front approach allows one to define purely intrinsic spatial densities, which have probabilistic interpretation, see Refs. [7–10,14]. However, the corresponding densities are obtained only as two-dimensional distributions in the impact parameter space. The relationship between these densities and the nonrelativistic three-dimensional distributions in the Breit frame in terms of the Abel transform was studied in Refs. [15,16]. Alternatively, the phase-space approach [17–19] allows one to define fully relativistic and unambiguous spatial densities, which in contrast to the light-front ones are three dimensional. However, in view of their dependence on both coordinates and momenta, these densities cannot have a strict probabilistic interpretation. The traditional Breit frame densities can also be obtained from a phase-space perspective by setting the system in the rest frame.

In the recent work of Ref. [20], the proper definition of the three-dimensional charge density has been revisited for a spin-0 system.[1] Closely following the logic of Ref. [11], the charge density possessing the usual probabilistic interpretation has been defined in the zero average momentum frame (ZAMF) of the system by using spherically symmetric sharply localized wave packets without invoking any approximations.[2] The definition has also been

[1]We thank Cedric Lorcé for pointing out that similar results have been published long ago in Ref. [21].

[2]Under ZAMF we mean a Lorentz frame in which the expectation value of the three-momentum for the state, specified by the given packet, is zero. For wave packets with a sharp localization around an eigenstate of the four-momentum operator, the ZAMF coincides with the rest frame of the system.

generalized to moving frames, and it was shown that in the infinite-momentum frame (IMF), the charge density turns into the well-known two-dimensional distribution in the transverse plane times the delta function in the longitudinal direction.

In the current paper we work out the details of the new definition of the electromagnetic densities for spin-1/2 systems. Analogously to the spin-0 case, we consider sharply localized wave packets and obtain local spatial distributions for the ZAMF as well as for moving frames.

Our work is organized as follows, in Sec. II we consider the spatial distributions of a spin-1/2 system in the ZAMF. Section III is devoted to the moving frames. In Sec. IV, we discuss the interpretation of the novel density distributions and comment on the proton radius controversy. We end with a summary in Sec. V.

II. THE ELECTROMAGNETIC DENSITIES OF A SPIN-1/2 SYSTEM IN THE ZAMF

We start with considering the electromagnetic densities of a spin-1/2 system in its ZAMF. We choose the four-momentum eigenstates $|p,s\rangle$ characterizing our system to be normalized as

$$\langle p',s'|p,s\rangle = 2E(2\pi)^3\delta_{s's}\delta^{(3)}(\mathbf{p}'-\mathbf{p}), \tag{1}$$

where $p=(E,\mathbf{p})$, with $E=\sqrt{m^2+\mathbf{p}^2}$ and m the particle's mass. These states are also eigenstates of the charge operator given by $\hat{Q}=\int d^3r\hat{j}^0(\mathbf{r},0)$, where $\hat{j}^\mu(\mathbf{r},0)$ is the electromagnetic current operator at $t=0$ in the Heisenberg picture.

The matrix elements of the electromagnetic current operator between momentum eigenstates of a spin-1/2 system can be parametrized in terms of two form factors:

$$\langle p',s'|\hat{j}^\mu(\mathbf{r},0)|p,s\rangle = e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\bar{u}(p',s')\left[\gamma^\mu F_1(q^2)+\frac{1}{2}i\sigma^{\mu\nu}q_\nu F_2(q^2)\right]u(p,s), \tag{2}$$

with p and s (p' and s') denoting the momentum and the polarization of the initial (final) states, respectively. The Dirac spinors are normalized as $\bar{u}(p,s')u(p,s)=2m\delta_{s's}$. The momentum transfer is given by $q=p'-p$, and $F_1(q^2)$ and $F_2(q^2)$ are the Dirac and the Pauli form factors, respectively. These are normalized as $F_1(0)=1$ and $F_2(0)=\kappa/m$, with κ being the anomalous magnetic moment of the spin-1/2 particle. We point out the difference in the normalization of the Pauli form factor compared to the literature.[3]

To define local electromagnetic densities we calculate the matrix element of the current operator in a state localized in coordinate space and take the size of the localization much smaller than all length scales characterizing the system. We consider a normalizable Heisenberg-picture state written in terms of a wave packet as

$$|\Phi,\mathbf{X},s\rangle = \int\frac{d^3p}{\sqrt{2E(2\pi)^3}}\phi(s,\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{X}}|p,s\rangle, \tag{3}$$

where from the normalization condition it follows that

$$\int d^3p|\phi(s,\mathbf{p})|^2=1. \tag{4}$$

Notice that the state $|\Phi,\mathbf{X},s\rangle$ depends on the spatial translation vector $\mathbf{X}$, whose interpretation will be discussed later. We *define* the density distributions in the ZAMF of the system by employing spherically symmetric wave packets with spin-independent profile functions $\phi(s,\mathbf{p})=\phi(\mathbf{p})=\phi(|\mathbf{p}|)$. For nonsymmetric wave packets, the newly defined electromagnetic densities can generally not be made independent of the profile function. However, the spherically symmetric choice of the wave packet appears legitimate given the absence of a preferred direction in the ZAMF. For later convenience, we introduce a dimensionless profile function $\tilde{\phi}$ via

$$\phi(\mathbf{p})=R^{3/2}\tilde{\phi}(R\mathbf{p}), \tag{5}$$

where R characterizes the size of the wave packet such that sharp localization corresponds to small values of R. The current density distribution for the state defined in Eq. (3) takes the form

$$\langle\Phi,\mathbf{X},s'|\hat{j}^\mu(\mathbf{x},0)|\Phi,\mathbf{X},s\rangle = \int\frac{d^3p\,d^3p'}{(2\pi)^3\sqrt{4EE'}}\bar{u}(p',s')\left[\gamma^\mu F_1(q^2)+\frac{i\sigma^{\mu\nu}q_\nu}{2}F_2(q^2)\right]u(p,s)\phi^\star(\mathbf{p}')\phi(\mathbf{p})e^{i\mathbf{q}\cdot(\mathbf{X}-\mathbf{x})}. \tag{6}$$

Introducing $\mathbf{r}=\mathbf{x}-\mathbf{X}$, the current density distribution takes the form

[3] The factor $1/m$ in front of the form factor F_2 appearing in commonly used parameterization mixes the orders of the $1/m$ expansion used below.

$$j^{\mu}_{\phi}(\mathbf{r}) \equiv \langle \Phi, \mathbf{0}, s' | \hat{j}^{\mu}(\mathbf{r}, 0) | \Phi, \mathbf{0}, s \rangle$$
$$= \int \frac{d^3P d^3q}{(2\pi)^3 \sqrt{4EE'}} \bar{u}(p', s') \left[\gamma^{\mu} F_1((E-E')^2 - \mathbf{q}^2) + \frac{i\sigma^{\mu\nu} q_{\nu}}{2} F_2((E-E')^2 - \mathbf{q}^2)\right] u(p, s) \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^{\star}\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad (7)$$

where $\mathbf{P} = (\mathbf{p}' + \mathbf{p})/2$, $E = \sqrt{m^2 + \mathbf{P}^2 - \mathbf{P}\cdot\mathbf{q} + \mathbf{q}^2/4}$ and $E' = \sqrt{m^2 + \mathbf{P}^2 + \mathbf{P}\cdot\mathbf{q} + \mathbf{q}^2/4}$.

The standard definition of the current density in terms of the form factors in the Breit frame, $F_i(q^2) = F_i(-\mathbf{q}^2)$, which we will refer to as "naive" following the terminology of Ref. [11], is obtained by first approximating the integrand in Eq. (7) by the two leading terms in the $1/m$ expansion and subsequently localizing the wave packet by taking the limit $R \to 0$ [11,12]. The $1/m$ expansion of the integrand leads to the following expressions:

$$J^0_{\phi,\mathrm{naive}}(\mathbf{r}) \equiv \int \frac{d^3P d^3q}{(2\pi)^3} \left\{F_1(-\mathbf{q}^2) - F_2(-\mathbf{q}^2)\left[\frac{\mathbf{q}^2}{4m} - \frac{i}{2m}\mathbf{q}\cdot(\mathbf{P}\times\boldsymbol{\sigma})\right]\right\} \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^{\star}\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}},$$
$$\mathbf{J}_{\phi,\mathrm{naive}}(\mathbf{r}) \equiv \frac{1}{m} \int \frac{d^3P d^3q}{(2\pi)^3} \left[\mathbf{P} F_1(-\mathbf{q}^2) - \frac{F_1(-\mathbf{q}^2) + mF_2(-\mathbf{q}^2)}{2} i\mathbf{q}\times\boldsymbol{\sigma}\right] \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^{\star}\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}}. \quad (8)$$

Here and in what follows, we use J^{μ} instead of j^{μ} to indicate that these densities are written as operators in spin space rather than the corresponding matrix elements. The $R \to 0$ limit of the expressions in Eq. (8) can be calculated without specifying the form factors and the profile function $\phi(\mathbf{p})$ using the methods developed to analyze loop integrals in quantum field theory [22,23]. For $F_1(q^2)$ and $F_2(q^2)$ decreasing at large q^2 faster than $1/q^2$ and $1/(q^2)^2$, respectively, by using the method of dimensional counting [22], the only nonvanishing contribution to $J^{\mu}_{\phi,\mathrm{naive}}(\mathbf{r})$ in the $R \to 0$ limit is obtained by substituting $\mathbf{P} = \tilde{\mathbf{P}}/R$, expanding the integrands in Eq. (8) in R around $R = 0$ and keeping up to the zeroth order terms. Doing so we obtain

$$J^0_{\mathrm{naive}}(\mathbf{r}) \equiv \lim_{R\to 0} J^0_{\phi,\mathrm{naive}}(\mathbf{r}) = \int \frac{d^3\tilde{P} d^3q}{(2\pi)^3} \left\{F_1(-\mathbf{q}^2) - F_2(-\mathbf{q}^2)\left[\frac{\mathbf{q}^2}{4m} - \frac{1}{R}\frac{i}{2m}\mathbf{q}\cdot(\tilde{\mathbf{P}}\times\boldsymbol{\sigma})\right]\right\} |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{-i\mathbf{q}\cdot\mathbf{r}}$$
$$= \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} G_E(-\mathbf{q}^2)$$
$$\equiv \rho^{\mathrm{ch}}_{\mathrm{naive}}(r),$$
$$\mathbf{J}_{\mathrm{naive}}(\mathbf{r}) \equiv \lim_{R\to 0} \mathbf{J}_{\phi,\mathrm{naive}}(\mathbf{r}) = \int \frac{d^3\tilde{P} d^3q}{(2\pi)^3} \left\{\frac{\tilde{\mathbf{P}}}{mR} F_1(-\mathbf{q}^2) - \frac{i\mathbf{q}\times\boldsymbol{\sigma}}{2m}[F_1(-\mathbf{q}^2) + mF_2(-\mathbf{q}^2)]\right\} |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{-i\mathbf{q}\cdot\mathbf{r}}$$
$$= \frac{\nabla_{\boldsymbol{r}}\times\boldsymbol{\sigma}}{2m} \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} G_M(-\mathbf{q}^2)$$
$$\equiv \nabla_{\boldsymbol{r}} \times \mathbf{M}_{\mathrm{naive}}(r), \quad (9)$$

where we employed the condition imposed on $\tilde{\phi}$ in Eq. (4) and used the fact that the integral over $\tilde{\mathbf{P}}$ of an odd function vanishes. Here, $G_E(q^2)$ and $G_M(q^2)$ refer to the Sachs electric and magnetic form factors [2] (note again the unconventional normalization of the Pauli form factor):

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4m} F_2(q^2),$$
$$G_M(q^2) = F_1(q^2) + mF_2(q^2). \quad (10)$$

Owing to the classical expression for the magnetization current $\mathbf{J}_{\mathrm{mag}}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{M}(\mathbf{r})$, the vector-valued function $\mathbf{M}_{\mathrm{naive}}(r)$ (an operator in the spin space) is interpreted as the magnetization distribution of the spin-1/2 particle. In the ZAMF of a spherically symmetric system, the spin operator is the only available vector, and the function $\mathbf{M}_{\mathrm{naive}}(r)$ is thus interpreted as the density of magnetic dipoles. It is usually expressed in terms of a scalar magnetization density $\rho^{\mathrm{mag}}_{\mathrm{naive}}(r)$, which corresponds to the Fourier transform of the magnetic form factor G_M in the Breit frame

$$\mathbf{M}_{\mathrm{naive}}(r) = \frac{1}{2m} \boldsymbol{\sigma} \rho^{\mathrm{mag}}_{\mathrm{naive}}(r),$$
$$\rho^{\mathrm{mag}}_{\mathrm{naive}}(r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} G_M(-\mathbf{q}^2). \quad (11)$$

In the notation we employ, the charge and scalar magnetization densities $\rho^{\mathrm{ch}}_{\mathrm{naive}}(r)$ and $\rho^{\mathrm{mag}}_{\mathrm{naive}}(r)$ are normalized to 1 and to the magnetic moment $\mu = 1 + \kappa$ of the particle, respectively. These two scalar densities incorporate the complete information about the internal structure of a composite spin-1/2 particle encoded in the electromagnetic form factors.

Notice that the final expressions in Eq. (9) do not depend on the shape of the wave packet (even if we would not require the spherical symmetry for the profile function). They correspond to the leading approximation to the spatial densities for packets localized with R much bigger than the Compton wavelength $1/m$ while much smaller than any other length scale characterizing the system. Such a definition, however, becomes doubtful for systems like light hadrons, whose characteristic length scales are comparable to or smaller than the Compton wavelength, see Ref. [11] and references therein.

An alternative definition of spatial densities, applicable to any systems, is obtained by localizing the wave packet describing the particle under consideration without performing a nonrelativistic expansion for the integrand in Eq. (7). Using the method of dimensional counting, the $R \to 0$ limit in Eq. (7) can, in fact, be taken for arbitrary (nonvanishing) values of the particle's mass as discussed in Ref. [20]. Doing so we obtain

$$
\begin{aligned}
J^0_\phi(\mathbf{r}) &= \int \frac{d^3\tilde{P} d^3 q}{(2\pi)^3} \left\{ F_1[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2] + \frac{i}{2}\mathbf{q}\cdot(\hat{\tilde{\mathbf{P}}}\times\boldsymbol{\sigma})F_2[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2] \right\} |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{-i\mathbf{q}\cdot\mathbf{r}}, \\
\mathbf{J}_\phi(\mathbf{r}) &= \int \frac{d^3\tilde{P} d^3 q}{(2\pi)^3} \left\{ \hat{\tilde{\mathbf{P}}} F_1[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2] - \frac{i}{2}(\mathbf{q}\times\boldsymbol{\sigma} - \mathbf{q}\times\hat{\tilde{\mathbf{P}}}\boldsymbol{\sigma}\cdot\hat{\tilde{\mathbf{P}}})F_2[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2] \right\} |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{-i\mathbf{q}\cdot\mathbf{r}},
\end{aligned}
\tag{12}
$$

where $\hat{\tilde{\mathbf{P}}} = \tilde{\mathbf{P}}/\tilde{P}$, with $\tilde{P} \equiv |\tilde{\mathbf{P}}|$. For spherically symmetric wave packets with $\tilde{\phi}(\tilde{\mathbf{P}}) = \tilde{\phi}(|\tilde{\mathbf{P}}|)$, the integration over $\tilde{\mathbf{P}}$ can be easily performed using Eq. (4). Denoting $\alpha = \cos\theta$ with θ being the angle between the vectors $\tilde{\mathbf{P}}$ and $\mathbf{q}$, we obtain

$$
\begin{aligned}
J^0(\mathbf{r}) &= \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} d\alpha \frac{1}{2} F_1[(\alpha^2-1)\mathbf{q}^2] \equiv \rho_1(r), \\
\mathbf{J}(\mathbf{r}) &= \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} \frac{d\alpha}{2} \left\{ \alpha\hat{\mathbf{q}} F_1[(\alpha^2-1)\mathbf{q}^2] - \frac{i}{4}\mathbf{q}\times\boldsymbol{\sigma}(1+\alpha^2)F_2[(\alpha^2-1)\mathbf{q}^2] \right\} \\
&= \frac{\nabla_r \times \boldsymbol{\sigma}}{2m} \int \frac{d^3 q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} d\alpha \frac{1}{4}(1+\alpha^2) m F_2[(\alpha^2-1)\mathbf{q}^2] \\
&\equiv \nabla_r \times \mathbf{M}(r) \equiv \frac{\nabla_r \times \boldsymbol{\sigma}}{2m} \rho_2(r).
\end{aligned}
\tag{13}
$$

Notice that the first term in the curly brackets, being an odd function in α, does not contribute to the density after integrating over α. The spatial densities $J^\mu(\mathbf{r})$ defined above are unambiguously expressed in terms of the experimentally measurable form factors $F_1(q^2)$ and $F_2(q^2)$ meaning that for the choice of the spherically symmetric wave packet they do not depend on the specific functional form of the profile function and further details of the wave packet. However, in contrast to the naive definition in Eq. (9), the validity of Eq. (13) does not depend on the relationship between the Compton wavelength $1/m$ and other length scales characterizing the system. In particular, Eq. (13) can be also used to define the spatial densities of light hadrons.

It is striking that the obtained result for $J^\mu(\mathbf{r})$, expressed in terms of the form factors $F_1(q^2)$ and $F_2(q^2)$, does not depend on the particle's mass, similarly to the charge density for a spinless system introduced in Ref. [20]. This implies that the traditional expression for the density, $J^\mu_{\rm naive}(\mathbf{r})$, does *not* emerge from $J^\mu(\mathbf{r})$ by expanding about the static limit. As explained in Ref. [20] for the case of a spin-0 particle, this counterintuitive feature originates from the noncommutativity of the $R \to 0$ and $m \to \infty$ limits of $J^\mu_\phi(\mathbf{r})$ in Eq. (7). Notice that while a finite-order approximation in the $1/m$-expansion is valid when calculating the form factors in Eq. (2) provided $-q^2 \ll m^2$, its validity is violated in certain momentum regions when performing the integration in Eq. (7) if R is taken of the order of the Compton wavelength or smaller.

The ZAMF expression for the charge density $\rho_1(r)$ in Eq. (13) coincides with that of a scalar particle discussed in Ref. [20]. Given that the state $|\Phi, \mathbf{x}, s\rangle$ is an eigenstate of the charge operator with an eigenvalue 1, both $\rho_1(r)$ and the naive charge density $\rho^{\rm ch}_{\rm naive}(r)$ are normalized to 1. On the other hand, $\rho_2(r)$ is normalized to $\frac{2}{3}\kappa$ rather than to the magnetic moment $1+\kappa$ as it is the case for the naive magnetization density $\rho^{\rm mag}_{\rm naive}(r)$. We will come back to this point in Sec. III, where a geometrical interpretation of the obtained ZAMF densities will be given in terms of the densities defined in the infinite-momentum frame. To facilitate such an interpretation, it is instructive to rewrite $J^\mu(\mathbf{r})$, specified in Eq. (13), as

$$J^\mu(\mathbf{r}) = \frac{1}{4\pi}\int d\hat{\mathbf{n}} J^\mu_{\hat{\mathbf{n}}}(\mathbf{r}), \tag{14}$$

where $\hat{\mathbf{n}} \equiv \mathbf{n}/|\mathbf{n}|$ is a unit vector along the direction of the vector $\tilde{\mathbf{P}}$ in Eq. (12) and

$$J^0_{\hat{\mathbf{n}}}(\mathbf{r}) = \rho_{1,\hat{\mathbf{n}}}(\mathbf{r}), \qquad \mathbf{J}_{\hat{\mathbf{n}}}(\mathbf{r}) = \frac{1}{2m}\boldsymbol{\nabla}_{\mathbf{r}} \times \boldsymbol{\sigma}_\perp \rho_{2,\hat{\mathbf{n}}}(\mathbf{r}), \tag{15}$$

where

$$\rho_{i,\hat{\mathbf{n}}}(\mathbf{r}) = \rho_i(r_\perp)\delta(r_\parallel), \tag{16}$$

and the two-dimensional auxiliary densities $\rho_i(r_\perp)$ are defined in terms of the form factors $F_i(q^2)$ via

$$\begin{aligned}\rho_1(r_\perp) &= \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp} F_1(-\mathbf{q}_\perp^2),\\ \rho_2(r_\perp) &= \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp} m F_2(-\mathbf{q}_\perp^2).\end{aligned} \tag{17}$$

Here and in what follows, $\mathbf{a}_\parallel \equiv \mathbf{a}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}$ ($\mathbf{a}_\perp \equiv \mathbf{a} - \mathbf{a}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}} = \hat{\mathbf{n}}\times(\mathbf{a}\times\hat{\mathbf{n}})$) denote the component of a vector $\mathbf{a}$ parallel (perpendicular) to the vector $\mathbf{n}$, and $a_\parallel \equiv |\mathbf{a}_\parallel|$, $a_\perp \equiv |\mathbf{a}_\perp|$. Notice that the auxiliary densities $\rho_{i,\hat{\mathbf{n}}}(\mathbf{r})$ depend on the direction $\hat{\mathbf{n}}$ only through the arguments $r_\perp$ and $r_\parallel$ of the corresponding two-dimensional densities and the delta function.

III. THE ELECTROMAGNETIC DENSITIES OF A SPIN-1/2 SYSTEM IN MOVING FRAMES

To generalize the expressions in Eqs. (13) and (14) to moving frames, we follow the procedure of Ref. [20] and replace the packet in Eq. (7) with its boosted expression. Using Eq. (3) and the transformation properties of the momentum eigenstates under a boost $\Lambda_{\mathbf{v}}$ with velocity $\mathbf{v}$,

$$|p,s\rangle \xrightarrow{\Lambda_{\mathbf{v}}} U(\Lambda_{\mathbf{v}})|p,s\rangle = \sum_{s_1} D_{s_1 s}\left[W\left(\Lambda_{\mathbf{v}}, \frac{\mathbf{p}}{m}\right)\right]|\Lambda_{\mathbf{v}} p, s_1\rangle, \tag{18}$$

where $D_{s_1 s}[W]$ is a spin-1/2 representation of the corresponding Wigner rotation W [24], we express a normalizable Heisenberg-picture state located at the origin of a moving frame in terms of the spherically symmetric ZAMF quantity $\phi(\mathbf{p})$ as (see also Ref. [25]):

$$|\Phi,\mathbf{X},s\rangle_{\mathbf{v}} = \int \frac{d^3 p}{\sqrt{2E(2\pi)^3}}\sqrt{\gamma\left(1-\frac{\mathbf{v}\cdot\mathbf{p}}{E}\right)}\phi(\Lambda_{\mathbf{v}}^{-1}\mathbf{p})\sum_{s_1} D_{s_1 s}\left[W\left(\Lambda_{\mathbf{v}}, \frac{\Lambda_{\mathbf{v}}^{-1}\mathbf{p}}{m}\right)\right] e^{-i\mathbf{p}\cdot\mathbf{X}}|p,s_1\rangle, \tag{19}$$

where $\gamma = (1-v^2)^{-1/2}$, $E = \sqrt{m^2+\mathbf{p}^2}$, $\Lambda_{\mathbf{v}}$ is the Lorentz boost from the ZAMF to the moving frame and $\Lambda_{\mathbf{v}}^{-1}\mathbf{p} = \hat{\mathbf{v}}\times(\mathbf{p}\times\hat{\mathbf{v}}) + \gamma(\mathbf{p}\cdot\hat{\mathbf{v}} - vE)\hat{\mathbf{v}}$.

Analogously to the case of the ZAMF, using the method of dimensional counting we conclude that the only nonvanishing contribution to the electromagnetic charge densities can be obtained by substituting $\mathbf{P} = \tilde{\mathbf{P}}/R$, so that in the $R \to 0$ limit for the moving frame we obtain

$$\begin{aligned}J^0_{\phi,\mathbf{v}}(\mathbf{r}) &= \int \frac{d^3\tilde{P} d^3 q}{(2\pi)^3}\gamma(1-\mathbf{v}\cdot\hat{\tilde{\mathbf{P}}})\left\{F_1[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2] + \frac{1}{4}(\mathbf{q}\cdot\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma} - \hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma}\mathbf{q}\cdot\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}})F_2[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2]\right\}|\tilde{\phi}(\tilde{\mathbf{P}}')|^2 e^{-i\mathbf{q}\cdot\mathbf{r}},\\ \mathbf{J}_{\phi,\mathbf{v}}(\mathbf{r}) &= \int \frac{d^3\tilde{P} d^3 q}{(2\pi)^3}\gamma(1-\mathbf{v}\cdot\hat{\tilde{\mathbf{P}}})\left\{\frac{1}{2}(\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma} + \hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma}\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}})F_1[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2]\right.\\ &\quad \left. -\frac{i}{4}\boldsymbol{q}\times(\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}} - \hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma}\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\tilde{\mathbf{P}}}\cdot\boldsymbol{\sigma})F_2[(\hat{\tilde{\mathbf{P}}}\cdot\mathbf{q})^2 - \mathbf{q}^2]\right\}|\tilde{\phi}(\tilde{\mathbf{P}}')|^2 e^{-i\mathbf{q}\cdot\mathbf{r}},\end{aligned} \tag{20}$$

where we have introduced $\tilde{\mathbf{P}}' = \hat{\mathbf{v}}\times(\tilde{\mathbf{P}}\times\hat{\mathbf{v}}) + \gamma(\tilde{\mathbf{P}}\cdot\hat{\mathbf{v}} - v\tilde{P})\hat{\mathbf{v}}$ and a unit vector $\hat{\mathbf{m}} \equiv \hat{\tilde{\mathbf{P}}}'$. Furthermore, $\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}$ refers to the Wigner rotated spin operator

$$\boldsymbol{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}} \equiv D^\dagger[W(\Lambda_{\mathbf{v}},\hat{\mathbf{m}})]\boldsymbol{\sigma}D[W(\Lambda_{\mathbf{v}},\hat{\mathbf{m}})], \tag{21}$$

with

$$D[W(\Lambda_{\mathbf{v}},\hat{\mathbf{m}})] = \lim_{R\to 0} D\left[W\left(\Lambda_{\mathbf{v}}, \frac{\Lambda_{\mathbf{v}}^{-1}(\tilde{\mathbf{P}}/R)}{m}\right)\right]. \tag{22}$$

Next, we change the integration variable $\tilde{\mathbf{P}} \to \tilde{\mathbf{P}}'$ and define a vector valued function

$$\mathbf{n}(\mathbf{v},\hat{\mathbf{m}}) = \hat{\mathbf{v}} \times (\hat{\mathbf{m}} \times \hat{\mathbf{v}}) + \gamma(\hat{\mathbf{m}}\cdot\hat{\mathbf{v}} + v)\hat{\mathbf{v}}. \tag{23}$$

Given that $\tilde{\mathbf{P}} = \hat{\mathbf{v}} \times (\tilde{\mathbf{P}}' \times \hat{\mathbf{v}}) + \gamma(\tilde{\mathbf{P}}'\cdot\hat{\mathbf{v}} + v\tilde{P}')\hat{\mathbf{v}}$, it follows that $\hat{\mathbf{n}} = \hat{\tilde{\mathbf{P}}}$. It is easy to verify that the Jacobian of the change of variables $\tilde{\mathbf{P}} \to \tilde{\mathbf{P}}'$ cancels the first factor in the integrands in Eq. (20), leading to

$$\begin{aligned} J^0_{\phi,\mathbf{v}}(\mathbf{r}) &= \int \frac{d\hat{\mathbf{m}} d\tilde{P}' d^3q}{(2\pi)^3} |\tilde{\phi}(\tilde{\mathbf{P}}')|^2 e^{-i\mathbf{q}\cdot\mathbf{r}} \left\{ F_1[(\hat{\mathbf{n}}\cdot\mathbf{q})^2 - \mathbf{q}^2] + \frac{1}{4}(\mathbf{q}\cdot\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} - \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\mathbf{q}\cdot\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}})F_2[(\hat{\mathbf{n}}\cdot\mathbf{q})^2 - \mathbf{q}^2] \right\}, \\ \mathbf{J}_{\phi,\mathbf{v}}(\mathbf{r}) &= \int \frac{d\hat{\mathbf{m}} d\tilde{P}' d^3q}{(2\pi)^3} |\tilde{\phi}(\tilde{\mathbf{P}}')|^2 e^{-i\mathbf{q}\cdot\mathbf{r}} \left\{ \frac{1}{2}(\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} + \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}})F_1[(\hat{\mathbf{n}}\cdot\mathbf{q})^2 - \mathbf{q}^2] \right. \\ &\quad \left. - \frac{i}{4}\boldsymbol{q} \times (\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}} - \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})F_2[(\hat{\mathbf{n}}\cdot\mathbf{q})^2 - \mathbf{q}^2] \right\}. \end{aligned}$$

Using the spherical symmetry of $\tilde{\phi}(\tilde{\mathbf{P}}') = \tilde{\phi}(|\tilde{\mathbf{P}}'|)$ and Eq. (4), the integration over $\tilde{P}'$ becomes trivial. To keep the notation compact, we express the resulting densities in the form similar to Eq. (14):

$$J^\mu_{\mathbf{v}}(\mathbf{r}) = \frac{1}{4\pi}\int d\hat{\mathbf{m}} J^\mu_{\mathbf{v},\hat{\mathbf{m}}}(\mathbf{r}), \tag{24}$$

where

$$\begin{aligned} J^0_{\mathbf{v},\hat{\mathbf{m}}}(\mathbf{r}) &= \rho_{1,\hat{\mathbf{n}}}(\mathbf{r}) + \frac{i}{4m}\mathbf{\nabla}_{\mathbf{r}}\cdot[\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}, \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}]_-\rho_{2,\hat{\mathbf{n}}}(\mathbf{r}), \\ \mathbf{J}_{\mathbf{v},\hat{\mathbf{m}}}(\mathbf{r}) &= \frac{1}{2}[\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}, \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}]_+\rho_{1,\hat{\mathbf{n}}}(\mathbf{r}) + \frac{1}{4m}\mathbf{\nabla}_{\mathbf{r}} \\ &\quad \times([\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}}, \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}]_-\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\rho_{2,\hat{\mathbf{n}}}(\mathbf{r}), \end{aligned} \tag{25}$$

with $[A,B]_\pm \equiv AB \pm BA$ and the auxiliary densities $\rho_{i,\hat{\mathbf{n}}}(\mathbf{r})$ being defined in Eq. (16) with the superscripts $\perp$ and $\parallel$ of various vectors referring, as before, to the direction of $\mathbf{n}$.

The above expressions simplify considerably in two extreme cases. First, in the particle's ZAMF with $v = 0$ and $\gamma = 1$, we have $\mathbf{n}(\mathbf{v},\hat{\mathbf{m}}) = \hat{\mathbf{m}}$ and $\mathbf{\Sigma}_{\mathbf{v},\hat{\mathbf{m}}} = \boldsymbol{\sigma}$. Thus, one can simply replace in Eq. (24) $\hat{\mathbf{m}}$ with $\hat{\mathbf{n}}$. Then, given Eq. (16), the integration over $\hat{\mathbf{n}}$ of the second (first) term on the right-hand side of the first (second) equality in Eq. (25) yields a vanishing result, so that one encounters the expressions in Eqs. (14)–(17).

The second interesting case corresponds to the infinite-momentum frame (IMF) with $v \to 1$ and $\gamma \to \infty$, for which the vector-valued function $\hat{\mathbf{n}}$ turns to $\hat{\mathbf{v}}$. Then, the integrand in Eq. (24) depends on $\hat{\mathbf{m}}$ only through the Wigner rotation matrices. Performing the integration over $\hat{\mathbf{m}}$, we obtain for the spatial densities in the IMF

$$\begin{aligned} J^0_{\mathrm{IMF}}(\mathbf{r}) &\equiv J^0_{\hat{\mathbf{v}}}(\mathbf{r}) = \rho_{1,\hat{\mathbf{v}}}(\mathbf{r}) + \hat{\mathbf{v}}\cdot\frac{\mathbf{\nabla}_{\mathbf{r}}\times\boldsymbol{\sigma}_\perp}{4m}\rho_{2,\hat{\mathbf{v}}}(\mathbf{r}), \\ \mathbf{J}_{\mathrm{IMF}}(\mathbf{r}) &\equiv \mathbf{J}_{\hat{\mathbf{v}}}(\mathbf{r}) = \frac{\mathbf{\nabla}_{\mathbf{r}}\times\boldsymbol{\sigma}_\perp}{4m}\rho_{2,\hat{\mathbf{v}}}(\mathbf{r}), \end{aligned} \tag{26}$$

where the densities $\rho_{i,\hat{\mathbf{v}}}(\mathbf{r})$ are given in Eqs. (16) and (17) with $\hat{\mathbf{n}} = \hat{\mathbf{v}}$. The appearance of $\delta(r_\parallel)$, see Eq. (16), reflects the fact that in the IMF, the system is Lorentz contracted to a two-dimensional object perpendicular to the velocity $\hat{\mathbf{v}}$ of the moving frame. It thus makes sense to introduce two-dimensional densities in the impact parameter space spanned by $\mathbf{r}_\perp$ by integrating the three-dimensional densities in Eq. (26) over $r_\parallel$:

$$\begin{aligned} \int dr_\parallel J^0_{\hat{\mathbf{v}}}(\mathbf{r}) &= \rho_1(r_\perp) + \hat{\mathbf{v}}\cdot\frac{\mathbf{\nabla}_{\mathbf{r}}\times\boldsymbol{\sigma}_\perp}{4m}\rho_2(r_\perp), \\ \int dr_\parallel \mathbf{J}_{\hat{\mathbf{v}}}(\mathbf{r}) &= \frac{\mathbf{\nabla}_{\mathbf{r}}\times\boldsymbol{\sigma}_\perp}{4m}\rho_2(r_\perp). \end{aligned} \tag{27}$$

Notice that the two-dimensional transverse charge and magnetization densities of spin-1/2 particles in the IMF have been extensively discussed in the literature, see, e.g., [26–28]. As for the expression in the last line of Eq. (27), the quantity $\frac{1}{2m}\rho_2(r_\perp)$ has been interpreted in Ref. [27] as a true magnetization density of the system, which generates the anomalous magnetic moment. On the other hand, Ref. [10] argued that a more natural interpretation of the anomalous magnetization density is provided by the distribution

$$\tilde{\rho}_2(\mathbf{r}_\perp) = -\frac{1}{2m} r_y \frac{\partial \rho_2(r_\perp)}{\partial r_y}, \tag{28}$$

where r_y is a component of $\mathbf{r}$ that is perpendicular to both $\mathbf{v}$ and $\boldsymbol{\sigma}_\perp$. Clearly, both densities $\frac{1}{2m}\rho_2(r_\perp)$ and $\tilde{\rho}_2(\mathbf{r}_\perp)$ are normalized to the anomalous magnetic moment of the particle.

IV. DISCUSSION AND INTERPRETATION

Before discussing the interpretation of the obtained results, it is useful to briefly summarize what has been achieved so far. Starting from a general matrix element of the electromagnetic current operator $\hat{j}^\mu(\mathbf{x},0)$ in a wave-packet state $|\Phi,\mathbf{X},s\rangle$ of a composite spin-$1/2$ system in the ZAMF, defined via a spherically symmetric profile function $\phi(\mathbf{p})$, we have taken the limit of a sharp localization, $R\to 0$ in order to remove the information about the wave packet, thereby obtaining the corresponding spatial density $J^\mu(\mathbf{r})$. The resulting expression in the first line of Eq. (13) relates the charge density $J^0(r)\equiv\rho_1(r)$ to the Dirac form factor $F_1(q^2)$, while the current distribution $\mathbf{J}(\mathbf{r})$ is expressed in terms of the scalar magnetization density $\rho_2(r)$ related to the Pauli form factor $F_2(q^2)$. These expressions for the charge and current densities differ from the conventional ones given in terms of the Fourier transform of the Sachs form factors in the Breit frame, see Eqs. (9) and (11). This raises questions about the uniqueness of our results, their relation to the conventional densities and physical interpretation, which will be addressed below. To keep the discussion as transparent as possible, we focus in the following on a spin-0 system considered in Ref. [20] (but the arguments hold for spin-$1/2$ particles as well). The charge densities $\rho(r)$ and $\rho_{\rm naive}(r)$ of Ref. [20] for a spinless system coincide with $\rho_1(r)$ and $\rho^{\rm ch}_{\rm naive}(r)$ of this paper if one sets $F_1(q^2)\equiv F(q^2)$ and $F_2(q^2)=0$.

To uncover the meaning of the charge distribution we need to identify the way it can be probed experimentally, and we usually think of elastic lepton scattering. For an infinitely heavy system described by a *static* charge distribution $\rho_{\rm naive}(r)$, there are no recoil effects, and the differential cross section in the single-photon approximation is given by the Mott cross section for scattering off a pointlike charge times $|F(-\mathbf{q}^2)|$, see, e.g., [29] for a textbook discussion. This exact result provides the means for directly accessing $\rho_{\rm naive}(r)$ experimentally, and it runs deep in our way of visualizing and interpreting the charge distribution of a composite system in nonrelativistic, quantum mechanical settings as common in atomic, nuclear, and solid-states physics. Notice that the static limit actually corresponds to the limit of $c\to\infty$, in which the Lorentz group reduces to the Galilean one, so that the charge distribution $\rho_{\rm naive}(r)$ becomes frame independent.

Clearly, the static limit is merely an idealization that can be imposed to approximate low-energy dynamics of composite systems. Contrary to what is sometimes claimed in the literature, see, e.g., [29], the Breit distribution $\rho_{\rm naive}(r)$ *cannot* be interpreted as the intrinsic charge density of the system beyond the strict static limit [20]. While it is possible to perturbatively (i.e., based on the $1/m$-expansion) take into account corrections beyond the static limit using an alternative definition of the charge density with the wave packet localized at distances well above its Compton wavelength [11], the resulting spatial distribution does not entirely reflect the internal structure of the system, being dependent on the wave packet. The usefulness of such a definition of the charge density thus relies on the Compton wavelength being much smaller than the size of the system as determined by the charge radius [11].

In contrast, the charge density defined using sharply localized spherically symmetric wave packets as done here and in Ref. [20] is valid for any massive system, regardless of the relationship between the Compton wavelength and the characteristic scale of the system related to the charge radius. Moreover, the resulting charge distribution $\rho(r)$ comes out to be independent of the particle's mass, which implies that $\rho_{\rm naive}(r)\neq\lim_{m\to\infty}\rho(r)$ and makes the static picture described above inappropriate for the interpretation of $\rho(r)$. More precisely, the large momentum components of the wave packet play a crucial role in the definition of the charge density $\rho(r)$, which thus represents an intrinsically relativistic quantity. Remarkably, imposing the sharp localization limit in the definition of the charge density forces one to think of $\rho(r)$ in a holographiclike picture in terms of a continuous superposition of images taken in all possible IMF [20]. The usual nonrelativistic interpretation of the rest-frame charge distribution of a heavy system characterized by the density $\rho(r)$ can only be obtained by reconstructing $\rho_{\rm naive}(r)$ from $\rho(r)$.

The above-mentioned holographiclike relationship between the ZAMF and IMF distributions also holds for the electromagnetic densities of a spin-$1/2$ particle considered in this paper. Comparing Eqs. (14) and (15) to Eq. (26), one observes that the spatial current densities $J^\mu(\mathbf{r})$ in the ZAMF are given by integrating the three-dimensional IMF densities $J^\mu_{\hat{\mathbf{v}}}(\mathbf{r})$ over all possible directions:

$$J^0(\mathbf{r})=\frac{1}{4\pi}\int d\hat{\mathbf{v}}J^0_{\hat{\mathbf{v}}}(\mathbf{r}),\quad \mathbf{J}(\mathbf{r})=2\times\frac{1}{4\pi}\int d\hat{\mathbf{v}}\mathbf{J}_{\hat{\mathbf{v}}}(\mathbf{r}). \tag{29}$$

This feature can be understood already by looking at the defining expressions for the densities in Eq. (7): taking the limit $R\to 0$ to remove the information about the spherically symmetric wave packet profile function from the definition of the densities brings the integrands in Eq. (7) to the kinematics with the total momentum $|\mathbf{P}|$ being larger than all other momentum scales, which in turn corresponds to the IMF (up to the Wigner rotations). In the IMF, the system is Lorentz-contracted to essentially a two-dimensional object perpendicular to the velocity of the moving frame with the densities given in Eq. (26). While only such two-dimensional images of the system are observed in the IMF, the expression in the ZAMF reconstructs the full three-dimensional structure by putting together all possible two-dimensional "images." In general, the full image of a d-dimensional object can be reconstructed by putting together its all possible $d-1$ dimensional images. This kind of representation is possible for all positive integers $d > 1$. We further emphasize that the extra factor of 2 that appears in the second equality in Eq. (29) has its origin in the Wigner rotations of spin states when performing Lorentz boosts to the IMF. From our point of view this picture makes intuitively clear why the spherical symmetry of the wave packet, in which the state is prepared, and its independence of the spin polarization play important role in the approach we used in this paper. Without these two assumptions the defined densities of Eq. (12) would depend on the information about the profile function $\phi(\mathbf{p}, s)$ of the packet, and hence definitely would not correspond to intrinsic distributions. Taking nonsymmetric packets would be analogous to trying to reconstruct a three-dimensional image of an object by putting together two-dimensional pictures from all possible directions taken with different coefficients of magnification for different directions. The resulting three-dimensional image would be distorted compared to the true image of the object. Notice here that charge densities defined in the light front [7–10,14] and the phase-space [17–19] approaches do not depend on any assumptions about the wave packets.

It is worth mentioning that the second moment of the charge distribution $\rho(r)$, the quantity that should be interpreted as the mean square charge radius of the system, is related to the form factor slope via[4]

$$\langle r^2 \rangle = 4F_1'(0), \tag{30}$$

where the prime denotes differentiation with respect to q^2. This is in contrast to the usual relationship

$$\langle r^2 \rangle_{\text{naive}} = 6\left(F_1'(0) + \frac{F_2(0)}{4m}\right), \tag{31}$$

motivated by the static definition $\rho^{\text{ch}}_{\text{naive}}(r)$. Thus, the size of, e.g., the proton measured by the new charge distribution $\rho_1(r)$ is $\sqrt{\langle r_{\text{p}}^2 \rangle} = 0.62649$ fm, which differs from $\sqrt{\langle r_{\text{p}}^2 \rangle_{\text{naive}}} = 0.8409(4)$ fm [30,31]. This discrepancy, however, has no practical implications since the radius extracted from electron-proton scattering as well as from electronic and muonic hydrogen is *defined* based on the expansion of the electric Sachs form factor around $q^2 = 0$,

$$G_E(q^2) = G_E(0)\left[1 + q^2 \frac{\langle r^2 \rangle}{6} + \ldots\right], \tag{32}$$

which is consistently used in the theory underlying these processes. What we have shown is that the radius defined via sharply localized packets, is indeed smaller due to the squeezing of the density already explained in Ref. [20]. This also applies to the magnetic radius related to the slope of the Pauli form factor at zero momentum transfer.

Finally, as the charge and magnetization densities are given in terms of $\mathbf{r} = \mathbf{x} - \mathbf{X}$, $\mathbf{X}$ should be interpreted as the position of the charge and magnetization centers of the system.

V. SUMMARY AND CONCLUSIONS

Using the prescription introduced in Ref. [20], we worked out the details of the new definition of spatial distributions of the electromagnetic current for composite spin-1/2 systems. We obtained relationships between the form factors and the local densities in the ZAMF and moving frames. Our definition is applicable to any system, irrespective of the relation between the Compton wavelength and other length scales characterizing the system. We have also worked out the expressions for the electromagnetic densities in moving frames including IMF. Similarly to the charge density of a spin-0 system studied in Ref. [20], the obtained electromagnetic spatial distributions possess a holographiclike interpretation in terms of the two-dimensional images made in all possible IMF.

We also explored an alternative way to define the spatial densities by employing the static approximation as suggested in Ref. [11], thereby recovering the conventional expressions in terms of the Fourier transform of the Sachs form factors in the Breit frame. Comparing the two definitions, we find that the static distributions cannot be obtained as a systematic approximation to our exact expressions. This is due to noncommutativity of the limits of an infinitely heavy system and a sharply localized packet.

ACKNOWLEDGMENTS

We thank Cédric Lorcé for useful comments on the manuscript. This work was supported in part by BMBF (Grant No. 05P18PCFP1), by DFG and NSFC through funds provided to the Sino-German CRC 110 "Symmetries and the Emergence of Structure in QCD" (NSFC Grant No. 11621131001, DFG Project-ID 196253076—TRR 110), by ERC NuclearTheory (Grant No. 885150) and ERC EXOTIC (Grant No. 101018170), by CAS through a President's International Fellowship Initiative (PIFI) (Grant No. 2018DM0034), by the VolkswagenStiftung (Grant No. 93562), and by the EU Horizon 2020 research and innovation programme (STRONG-2020, Grant Agreement No. 824093).

[4]The same expression was obtained in Ref. [12] for the mean square charge radius of the system defined in two-dimensional light-front dynamics.

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